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ABSTRACT

In this paper, we study the Hamiltonian dynamics of charged particles subject to a non-self-consistent stochastic electric field when the plasma is in the so-called weak turbulent regime. We show that the asymptotic limit of the Vlasov equation is a diffusion equation in the velocity space but homogeneous in the physical space. We obtain a diffusion matrix, quadratic with respect to the electric field, which can be related to the diffusion matrix of the resonance broadening theory and of the quasilinear theory, depending on whether the typical autocorrelation time of particles is finite or not. In the self-consistent deterministic case, we show that the asymptotic distribution function is homogenized in the space variables, while the electric field converges weakly to zero. We also show that the lack of compactness in time for the electric field is necessary to obtain a genuine diffusion limit. By contrast, the time compactness property leads to a "cheap" version of the Landau damping: the electric field converges strongly to zero, implying the vanishing of the diffusion matrix, while the distribution function relaxes, in a weak topology, toward a spatially homogeneous stationary solution of the Vlasov–Poisson system.

I. INTRODUCTION

Here, we are interested in a problem of particle diffusion, which is produced by the wave–particle interaction. In plasma physics, the wave–particle interaction is an important phenomenon, which stands at the root of Landau damping, of wave heating, of numerous instabilities, and of some regimes of anomalous transport in magnetically confined plasmas. This work is closely related to the so-called quasilinear (QL) theory, which describes the nonlinear relaxation of the weak warm beam-plasma instability through the derivation of a diffusion equation in the velocity variable conjugated with the prediction of an associated diffusion coefficient. This topic has led to a longstanding controversy that is not solved yet.[2,10,14,18,19,25–34,36,38,40–46,52,54,56,57] References cited above are not exhaustive but testify to the huge literature on this subject. For a brief history on the development of the QL theory, we refer the reader to Refs. 10 and 44. Furthermore, the QL diffusion coefficient is quite frequently used for modeling particle transport in different branches of plasma physics, such as laser-plasma interaction or magnetized plasma turbulence. Since the QL approximation is ubiquitous, particularly in kinetic modelling, it is then important to assess, in the most rigorous possible way, whether the QL theory is valid or not. A complete and rigorous proof of the QL theory goes beyond the purpose of this paper, which aims at taking stock of what can or cannot be rigorously proven at this time.

We now sketch this problem in dimension one (see, e.g., Ref. 34 for an intuitive introduction and Chap. 8 and 9 of Ref. 21 or Chap. 7 of Ref. 31 for a more exhaustive one). We consider a two-dimensional distribution function of particles in the two-dimensional phase space (x, v). This distribution function is initially given by a one-dimensional (in v) spatially uniform (in x) beam-plasma system. This beam corresponds to a gentle and small bump on the tail of the electronic plasma velocity distribution function (see, e.g., Sec. 9.4 of

Ref. 39). The study of the bump-on-the-tail instability dates back to the pioneering work of Buneman[16] on the two-stream instability, where each stream is considered as a mono-kinetic beam. Using the Nyquist method (see, e.g., Sec. 9.6 of Ref. 39), Penrose[49] derived a criterion, the so-called Penrose criterion for instability, under which the beam-plasma system distribution function (spatially uniform and one-dimensional in the velocity variable) constitutes an unstable equilibrium (i.e., an unstable stationary solution of the Vlasov–Poisson equations). Then, any initial small perturbations of the beam-plasma system are destabilized by the inversion of the electron population corresponding to the positive slope interval of the velocity distribution. This gives rise to electrostatic waves, which first grow linearly until the beginning of a saturation stage, where the amplitude of waves reaches a non-negligible value. In this resulting wave spectrum, the particle dynamics becomes chaotic enough in their range of phase velocities so that the bump is eroded with eventually a plateau formation in the distribution function. Simultaneously, there is a transfer of momentum from particles to electric waves, generating a turbulent spectrum of waves. This scenario was first predicted on a theoretical basis[26,57] by considering the wave–particle interaction as perturbative and neglecting all nonlinear wave–wave interactions in the Vlasov–Poisson equation, except for their effect on the space-averaged distribution function $f = f(t,v)$. This led to the set of QL equations coupling the distribution function f and the Fourier modes $E(t,k)$ of the electric field,

$$\partial_t f(t,v) - \partial_v(D_{QL}(t,v)\partial_v f(t,v)) = 0, \tag{1}$$

$$\partial_t |E(t,k)|^2 = 2\gamma(t,k)|E(t,k)|^2, \tag{2}$$

where the QL diffusion coefficient is given by

$$D_{QL}(t,v) = \pi \sum_{k\in\mathbb{Z}} |E(t,k)|^2 \delta(\omega(t,k) - kv)^{61}. \tag{3}$$

The real functions $(t,k) \mapsto \omega(t,k)$ and $(t,k) \mapsto \gamma(t,k)$ satisfy the following dispersion equation:

$$\begin{aligned} &\mathbb{D}(k,\omega(t,k) + \mathrm{i}\gamma(t,k)) = 0, \quad \text{with} \\ &\mathbb{D}(k,\omega(t,k) + \mathrm{i}\gamma(t,k)) := 1 + \frac{\omega_p^2}{k^2}\int_{\mathbb{R}} dv \, \frac{k\partial_v f(t,v)}{\omega(t,k) - kv + \mathrm{i}\gamma(t,k)}, \end{aligned} \tag{4}$$

where ω_p is the plasma frequency.[21,26,39,57] System (1)–(4) is a closed and self-consistent system of equations. We recall that in the case of the gentle-bump-on-the-tail instability, an approximate solution of the dispersion equation is given (see, e.g., Ref. 39) by the so-called Bohm–Gross relation

$$\omega^2(t,k) \simeq \omega^2(k) := \omega_p^2(1 + 3k^2\lambda_D^2), \tag{5}$$

with the Debye length $\lambda_D := v_{th}/\omega_p$ and the thermal velocity squared $v_{th}^2 := \int f v^2 dv / \int f dv$. The approximate growth rate is given by

$$\gamma(t,k) \simeq \frac{\pi}{2}\frac{\omega_p^2}{k^2}\,\omega(k)\int_{\mathbb{R}} dv \, \delta(\omega(k) - kv)k\partial_v f(t,v). \tag{6}$$

Therefore, the system constituted by (1), (2), (5), and (6) is also a closed and self-consistent system. This approximate solution relies on the following assumptions: $v \mapsto f(t,v)$ is even, $\gamma/\omega \ll 1$ (weak instability), and $k\lambda_D \ll 1$ (long wavelength approximation) (see, e.g., Ref. 39 for more details). Let us note that the dispersion equation and its approximate solution are the same as for the Landau damping case, where the damping rate γ given by (6) is negative because the slope of f so is.

We must emphasize that, even from a physical and physicist's point of view, the derivation of quasilinear theory from either a deterministic or a probabilistic approach is actually not clear. Indeed, the original 1962 derivation,[26,57] briefly exposed above, is deterministic. Right after there were many other derivations of the QL theory, most of them (see, e.g., Refs. 3, 5, and 21 and references therein) appeal to some statistical arguments, like the random phase approximation (RPA), and invoke some time/space decorrelation hypotheses. From a numerical point of view, it has been shown in Ref. 10 that a statistical ensemble average of solutions of the Vlasov–Poisson system is required to recover a QL description of the long time behavior of the weak warm beam-plasma system.

In this work, we consider both the self-consistent deterministic case and the non-self-consistent stochastic case. Here, the term "self-consistent" means that the electric field is produced by the particles themselves through the coupling with the Poisson equation. In the self-consistent deterministic case, we show that the asymptotic distribution function is homogenized in the space variables, while the self-consistent electric field converges weakly to zero. As already observed in related works (e.g., Ref. 6), we show that the lack of time compactness for the electric field is compulsory to obtain a non-trivial and thus a diffusion limit for the Vlasov equation. By contrast, the time compactness property leads to a cheap version of the Landau damping, where the electric field converges strongly to zero (entailing a null diffusion matrix) and the distribution function converges weakly to a spatially homogeneous stationary solution of the Vlasov–Poisson system. Actually, the difficult part is to show a non-zero diffusion limit in the presence of fast time oscillations. Using a Duhamel formula, we formally derive a diffusion equation for the asymptotic distribution function, which depends only on the time and velocity variables. Unfortunately, we are not able to justify rigorously this diffusion limit. This task requires a new approach, which will be the matter of a future work. It is worthwhile to mention that in the nonlinear regime, the saturation of the weak warm beam-plasma instability generates in phase space a type of turbulence, which has a very close connection to Hamiltonian chaos theory (see, e.g., Refs. 8 and 35 and references therein). A complete treatment of the self-consistent deterministic case remains an open issue, the proof of which must be based at least (but not only) on the same ingredients than those used for proving the Landau damping[48] and more particularly on the control of nonlinear wave–wave interactions (e.g., plasma echoes). In contrast with Landau damping, the main and not the least difficulty is that perturbations are not arbitrarily small, since wave amplitudes are amplified by the instability. From a mathematical point of view, this makes the nonlinear wave–wave interactions more difficult to control, especially for showing that the latter remains negligible at least at the end of the relaxation process. The proof of QL diffusion for this "inverse landau damping" problem remains a challenge. Nevertheless, studying the non-self-consistent problem remains meaningful. Indeed, it was observed in numerical simulations of the self-consistent problem[10] that when the distribution function is enough phase-space homogenized (after quite a long time), the problem falls into the non-self-consistent framework, even in the strong nonlinear regime.

As explained in plasma physics literature (see, e.g., Refs. 3 and 21), diffusion in the QL theory comes from the time decorrelation property of the electric field, which can be considered as a random field. We then place ourselves in similar modeling hypotheses. This second framework is then closer to the case of particles evolving in a given bath of (random) waves[9,18,29–31] or particles subject to a reversible reflection law, which has convenient mixing properties.[6] As a result, we prove that the asymptotic limit of the Vlasov equation is a diffusion equation in the velocity space, where the diffusion matrix is given by the space–time autocorrelation function of the stochastic electric field, the so-called Reynolds electric stress tensor. Hence, the diffusion matrix is quadratic with respect to the electric field. By specializing a little bit more the structure of our electric field, we recover, at least from a formal point of view, the diffusion matrix predicted by the QL theory. Our diffusion matrix can also be related to a refinement of the QL theory called the resonance broadening theory.[1,21,27,51,58] For the present problem and to our knowledge, our results have not been found in the literature so far. For the proof of the non-self-consistent stochastic case, we follow the strategy introduced in Ref. 50, which relies on short-time decorrelation properties. These techniques have been successfully used in various physical contexts.[7,15,20,37,47]

The outline of this paper is as follows: Sec. II describes the weak turbulent regime, which is characterized by some dimensionless parameters. In Sec. III, we deal with the self-consistent deterministic case. In Sec. IV, we deal with the non-self-consistent stochastic case. Section IV A collects all the hypotheses on the stochastic electric field. Section IV B contains our main result about the diffusion limit of the Vlasov equation, the proof of which is done in Sec. IV C. Finally, Sec. IV D connects our result with some kinetic turbulence theories of plasma physics, such as the resonance broadening theory and the quasilinear theory.

II. THE WEAK TURBULENT REGIME

A. Dimensionless parameters

The Vlasov–Poisson system, describing the self-consistent evolution of the distribution function of particles $f = f(t,x,v)$ in an electrostatic plasma, reads

$$\partial_t f + v\cdot\nabla_x f + \frac{q}{m}E\cdot\nabla_v f = 0, \tag{7}$$

$$E = -\nabla\Phi, \quad -\Delta\Phi = \frac{q}{\varepsilon_0}\left(\int_{\mathbb{R}^d} dv\, f - 1\right). \tag{8}$$

Here, $t\in\mathbb{R}$, $x\in\mathbb{T}^d := (\mathbb{R}/2\pi\mathbb{Z})^d$, and $v\in\mathbb{R}^d$, represent, respectively, time, position, and velocity of particles of charge q and mass m, which are accelerated by the "turbulent" electric field $E = E(t,x)$. Since the plasma is globally neutral, we have

$$\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} dx \int_{\mathbb{R}^d} dv\, f = 1. \tag{9}$$

In order to have a well-posed problem, we must add the zero-mean electrostatic condition

$$\int_{\mathbb{T}^d} dx\, E = 0. \tag{10}$$

Indeed, condition (10) is necessary to invert the Laplacian operator Δ. The phase space is denoted by $Q := \mathbb{T}^d\times\mathbb{R}^d$. In order to write the Vlasov–Poisson system [(7) and (8)] in a dimensionless form, we need to introduce a time unit $\hat{t}$, a length unit $\hat{x}$, a velocity unit $\hat{v}$, and typical amplitudes $\hat{E}$, $\hat{\Phi}$ and $\hat{f}$ for the electric field, the electric potential, and the distribution function, respectively. The dimensionless variables and physical quantities read

$$t' = \frac{t}{\hat{t}}, \quad x' = \frac{x}{\hat{x}}, \quad v' = \frac{v}{\hat{v}}, \quad E' = \frac{E}{\hat{E}}, \quad \Phi' = \frac{\Phi}{\hat{\Phi}}, \quad f' = \frac{f}{\hat{f}}. \tag{11}$$

We set

$$\hat{n} := \hat{f}\hat{v}^d, \tag{12}$$

the typical value of the macroscopic (charge) density of particles. Using the Poisson equation (8), we obtain the following dimensional equation:

$$\hat{E} = \hat{n}\hat{x}q/\varepsilon_0. \tag{13}$$

In an electrostatic plasma, the typical length scale is the Debye length λ_D, while the typical velocity is the thermal velocity v_{th}. The plasma frequency ω_p, which is related to a typical fast oscillation time of an electrostatic plasma, is then given by

$$\omega_p = \frac{v_{th}}{\lambda_D}. \tag{14}$$

The typical electric and kinetic energies are, respectively, $\mathcal{E}_{el} = \varepsilon_0|\hat{E}|^2$ and $\mathcal{E}_{kin} = \hat{n}m\hat{v}^2$. The distribution function f has a typical evolution/relaxation time τ_{rel}, while the turbulent electric field has two time scales. A slow time scale τ_L is associated with the instantaneous growth or damping rate $\gamma_L := 1/\tau_L$ of the electric field, while a fast time scale is related to both the wave (electric field) autocorrelation time τ_{ac} and the particle autocorrelation time τ_D. The time τ_{ac} is the lapse of time needed for a resonant particle, traveling at the same velocity as the phase velocity of a typical wave, to cross the localized spatial extent of the oscillatory electric field disturbance. This time can also be seen as the time needed for a resonant particle to resolve the finite frequency width of the wave spectrum. In other words, the time τ_{ac} can be seen as the turnover or the lifetime of a typical wave measured or felt by a resonant particle traveling at the same velocity as the phase velocity of this wave. Then, the synchronization between a wave and a particle occurs in a lapse of time of the order of τ_{ac}, during which they interact by momentum transfer. The time τ_D is the autocorrelation or spreading time of particles, i.e., the lapse of time after which two close particles or orbits are completely separated from each other. In the plasma physics literature, the time τ_D is called the Dupree time.[27] A particle distribution function evaluated at two different times separated by a time interval of the order of τ_D is then decorrelated. The relaxation time τ_{rel} of the distribution function is then of the order of τ_D. In the self-consistent case, where the Poisson equation is used to compute the electric field from the particle distribution function, this implies that two evaluations in time of the electric field, separated by a time interval of the order of τ_D, are also decorrelated. We now set

$$\hat{t} := \tau_L, \quad \hat{x} := \lambda_D, \quad \hat{v} := v_{th}. \tag{15}$$

Let $\varepsilon \in (0,1)$ be a small dimensionless parameter and $\bar{\tau} \in [0,+\infty]$ be a positive dimensionless parameter, which may be finite or infinite. Then, the weak turbulence regime of an electrostatic plasma is defined by (see, e.g., Chap. 7 in Ref. 21)

$$\frac{\mathcal{E}_{el}}{\mathcal{E}_{kin}} = \frac{\varepsilon_0|\hat{E}|^2}{\hat{n}m\hat{v}^2} = \varepsilon, \quad \frac{1}{\omega_p\hat{t}} = \varepsilon^2, \quad \frac{\tau_{ac}}{\hat{t}} = \varepsilon^2, \quad \frac{\tau_D}{\hat{t}} = \bar{\tau}. \tag{16}$$

Using (11)–(16) and dropping the prime notation for dimensionless variables and physical quantities, we obtain from (7) and (8) the dimensionless Vlasov–Poisson equations

$$\partial_t f^\varepsilon + \frac{v}{\varepsilon^2}\cdot\nabla_x f^\varepsilon + \frac{E^\varepsilon}{\varepsilon}\cdot\nabla_v f^\varepsilon = 0, \quad (t,x,v) \in \mathbb{R}^+ \times \mathbb{T}^d \times \mathbb{R}^d, \tag{17}$$

$$E^\varepsilon = -\nabla\Phi^\varepsilon, \quad -\Delta\Phi^\varepsilon = \int_{\mathbb{R}^d} dv\, f^\varepsilon - 1. \tag{18}$$

The global neutrality condition (9) and the zero-mean electrostatic condition (10) keep the same. We just have to substitute f^ε to f in (9) and E^ε to E in (10).

B. Notation

In the rest of this paper, the notation $\langle\cdot,\cdot\rangle$ denotes the duality bracket between the space of distributions $\mathcal{D}'(\mathbb{R}^+ \times Q)$ and the space $\mathcal{D}(\mathbb{R}^+ \times Q)$ of indefinitely differentiable functions with compact support in $\mathbb{R}^+ \times Q$. The L^2-scalar product on the phase space $Q = \mathbb{T}^d \times \mathbb{R}^d$ is defined by

$$(f,g) := \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} dx \int_{\mathbb{R}^d} dv\, fg^*, \quad \forall f,g \in L^2(Q), \tag{19}$$

where the notation $(\cdot)^*$ stands for the complex conjugate. We then have, for $g \in L^1_{\text{loc}}(\mathbb{R}^+ \times Q)$,

$$\langle g, \varphi \rangle := \int_{\mathbb{R}^+} dt\, (g, \varphi), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^+ \times Q).$$

We denote the space average on the torus $\mathbb{T}^d$ by

$$\fint dx\, g(t, x, v) := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} dx\, g(t, x, v).$$

The one-parameter family of functions $\{g^\varepsilon\}_{\varepsilon>0}$, which we call sequences (respectively, subsequences) by abuse of language, must be understood as generalized sequences (respectively, subsequences) such as nets (respectively, subnets) in the sense of Moore–Smith or filters (respectively, finer filters) in the sense of Cartan (for more details, see, e.g., Refs. 13 and 59). We also use the notation $\overline{g^\varepsilon}$ to denote the cluster point, at least in the sense of distributions, of a family of functions $\{g^\varepsilon\}_{\varepsilon>0}$. We next define the free-flow operator by

$$\mathcal{L} := v \cdot \nabla_x.$$

We note $t \mapsto S^\varepsilon_t$ the group on $L^p(Q)$, $1 \le p \le \infty$, generated by the free-flow operator $\varepsilon^{-2}\mathcal{L}$. Then, an explicit formula for the group $t \mapsto S^\varepsilon_t$ is given by

$$(S^\varepsilon_t g)(x, v) = \exp\left(-\frac{t}{\varepsilon^2}\mathcal{L}\right) g(x, v) = g(x - vt/\varepsilon^2, v), \quad \forall g \in L^p(Q). \tag{20}$$

Eventually, the symbol $|\cdot|$ denotes either the modulus or the Euclidean norm depending on whether we deal with complex/real scalars or vectors.

III. THE SELF-CONSISTENT DETERMINISTIC CASE

In this section, we deal with the self-consistent deterministic case.

A. Ergodic theorem

We observe two orthogonal behaviors for the asymptotic limit of the Vlasov–Poisson system [(17) and (18)], depending on whether one makes or not the hypothesis of time compactness. The cornerstone of such observations is the ergodic property of the free-flow operator $\mathcal{L}$ on the torus, which we recall in the following lemma:

Lemma 1 (Ergodicity of the free flow on the torus). Let $g \in L^p(Q)$, with $1 \le p \le \infty$, satisfy

$$\mathcal{L}g = 0 \quad \text{in} \quad \mathcal{D}'(Q).$$

Then, $g = \fint dx\, g$ (g is independent of the variable x).

Proof. We first integrate the free-flow operator by using characteristic curves, and second, we use the spatial Fourier transform of the obtained solution. Indeed, the characteristic curves $(X(\tau), V(\tau))$ of the free flow satisfy the ODEs $\dot{X}(\tau) = V(\tau)$, $\dot{V}(\tau) = 0$, with initial conditions $X(0) = x_0$ and $V(0) = v_0$. Its solution is given by $(X(\tau) = x_0 + v_0\tau, V(\tau) = v_0)$. Since $(dg/d\tau)(X(\tau), V(\tau)) = (v \cdot \nabla_x g)(X(\tau), V(\tau)) = 0$, we obtain $g(X(\tau), V(\tau)) = g(X(0), V(0))$ or $g(x, v) = g(x - v\tau, v)$ for a.e. $(x, v) \in Q$ and all $\tau \in \mathbb{R}^+$. The Fourier transform in space of this last equation gives

$$(1 - \exp(ik \cdot v\tau))\hat{g}(k, v) = 0, \quad \forall k \in \mathbb{Z}^d,\ v \in \mathbb{R}^d,\ \tau \in \mathbb{R}.$$

This relation and $g \in L^p(Q)$, with $1 \le p \le \infty$ (g cannot be a Dirac mass in velocity), imply that the support of $\hat{g}$ is contained in the set $\text{supp}(\hat{g}) := \{(k, v) \in \mathbb{Z}^d \times \mathbb{R}^d \mid k \cdot v\tau \in 2\pi\mathbb{Z},\ \forall \tau \in \mathbb{R}\}$. For any $\delta, T, r, R > 0$ such that $\delta < T$ and $r < R$, the Lebesgue measure of the set $\text{supp}(\hat{g})$ for $\delta < \tau < T$ and $r < |v| < R$ is zero. This forces $\hat{g}$ to be equal to zero for all $k \ne 0$ and then $\text{supp}(\hat{g}) := \{(k, v) \mid k = 0,\ \ v \in \mathbb{R}^d\}$. Therefore, $g = \fint dx\, g$ and g is independent of the variable x. This ends the Proof of Lemma 1. □

Remark 1. The Proof of Lemma 1 is reminiscent of the Proof of Theorem 2.1 in Ref. 17, but it is not exactly the same. Indeed, here we do not use the Riemann–Lebesgue lemma (in the Fourier dual variable of v), whereas Ref. 17 does. More precisely, the Proof of Theorem 2.1 in Ref. 17, which is based on the weak formulation of the equation $g(x, v) = g(x - v\tau, v)$ against continuous compactly supported test functions, uses first the Fourier transform in the phase space Q to switch from real variables to Fourier variables in the weak formulation, and it uses second the Riemann–Lebesgue lemma together with the Lebesgue dominated convergence theorem to pass to the limit.

Based on the ergodicity of the free flow, we state the following theorem:

Theorem 1.1. *Let $\{f_0^\varepsilon\}_{\varepsilon>0}$ be a sequence of non-negative initial data and C_0 be a positive constant such that*

$$\|f_0^\varepsilon\|_{L^1(Q)} + \|f_0^\varepsilon\|_{L^\infty(Q)} \leq C_0, \quad \int_Q f_0^\varepsilon |v|^2\, dxdv \leq C_0, \quad \text{and}$$
$$\left\| E_0^\varepsilon := \nabla\Delta^{-1}\Big(\int_{\mathbb{R}^d} f_0^\varepsilon dv - 1\Big)\right\|_{L^2(\mathbb{T}^d)} \leq C_0.$$

Let $(f^\varepsilon, E^\varepsilon)_{\varepsilon>0}$ be a sequence of weak solutions of the Vlasov–Poisson system (17)–(18), with initial data $f^\varepsilon{}_{|_{t=0}} = f_0^\varepsilon$, the existence of which has been proved in Refs. 4, 12, 22, and 23 for all $\varepsilon > 0$. Then, we have the following:

(i) *There exists a function $f = f(t,v)$, independent of the variable x, such that $f \in L^\infty(\mathbb{R}^+; L^1 \cap L^\infty(\mathbb{R}^d))$, and up to subsequences, one has*

$$f^\varepsilon \rightharpoonup f \ \ in\ \ L^\infty(\mathbb{R}^+; L^\infty(Q)) \ \ weak-*,$$
$$⨍ dx\, f^\varepsilon \rightharpoonup f \ \ in\ \ L^\infty(\mathbb{R}^+; L^\infty(\mathbb{R}^d)) \ \ weak-*.$$

(ii) *The electric field E^ε converges weakly to zero as $\varepsilon \to 0$, more precisely,*

$$E^\varepsilon \rightharpoonup 0 \ \ in\ \ L^\infty(\mathbb{R}^+; W^{1,1+2/d}(\mathbb{T}^d)) \ \ weak-*.$$

(iii) *The expression*

$$\nabla_v \cdot ⨍ dx\, \frac{E^\varepsilon f^\varepsilon}{\varepsilon}$$

is uniformly (with respect to ε) bounded in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d)$; hence, up to a subsequence, it converges in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d)$, and we obtain

$$\partial_t f + \nabla_v \cdot \overline{⨍ dx\, \frac{E^\varepsilon f^\varepsilon}{\varepsilon}} = 0 \quad in\ \ \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d), \tag{21}$$
$$f_{|_{t=0}} = ⨍ dx\, f_0. \tag{22}$$

(iv) *Let $d \leq 4$. Moreover, if we suppose that there exists a constant κ, independent of ε such that*

$$\|E^\varepsilon\|_{W^s_{\rm loc}(\mathbb{R}^+; L^1(\mathbb{T}^d))} \leq \kappa, \ \ \text{with}\ s > 0, \quad \text{and} \quad \|\partial_t \Phi^\varepsilon\|_{L^1_{\rm loc}(\mathbb{R}^+; L^1(\mathbb{T}^d))} \leq \kappa, \tag{23}$$

then

$$⨍ dx\, \frac{E^\varepsilon f^\varepsilon}{\varepsilon} \rightharpoonup 0 \ \ in\ \ \mathcal{D}'([0,T] \times \mathbb{R}^d), \tag{24}$$
$$E^\varepsilon \to 0 \ \ in\ \ L^1([0,T] \times \mathbb{T}^d) \ \ \text{strong}$$

as $\varepsilon \to 0$, and (21) and (22) *degenerate into the following equations:*

$$\partial_t f = 0 \quad in\ \ \mathcal{D}'([0,T] \times \mathbb{R}^d), \tag{25}$$
$$f_{|_{t=0}} = ⨍ dx\, f_0. \tag{26}$$

Proof. Since $\|f_0^\varepsilon\|_{L^1(Q)} + \|f_0^\varepsilon\|_{L^\infty(Q)} \le C_0 < \infty$, by weak compactness arguments, there exists a function $f_0 \in L^1 \cap L^\infty(Q)$ such that f_0^ε (up to a subsequence) converges in $L^\infty(Q)$ weak-$*$ to f_0. Indeed, from $\|f_0^\varepsilon\|_{L^1(Q)} + \|f_0^\varepsilon\|_{L^\infty(Q)} \le C_0 < \infty$ and using standard weak compactness theorems, we obtain that there exists $f_0 \in \mathcal{M}_b \cap L^\infty(Q)$ such that $f_0^\varepsilon \rightharpoonup f_0$ in $\mathcal{M}_b \cap L^\infty(Q)$ weak$-*$. Here, $\mathcal{M}_b(Q)$ is the set of bounded measures on Q. Moreover, $\|f_0^\varepsilon\|_{L^1(Q)} + \|f_0^\varepsilon\|_{L^\infty(Q)} \le C_0 < \infty$ implies that $\|f_0^\varepsilon\|_{L^p(Q)} \le C_0 < \infty$, for $1 \le p \le \infty$. Therefore, we have also $f_0^\varepsilon \rightharpoonup f_0$ in $L^p(Q)$ weak$-*$ for $1 < p \le \infty$. This and the De La Vallé–Poussin theorem on the criterion for uniform equi-integrability implies that the family $\{f_0^\varepsilon\}_{\varepsilon>0}$ is uniformly equi-integrable. Finally, using Dunford–Pettis theorem, uniform equi-integrability implies that we also have $f_0^\varepsilon \rightharpoonup f_0$ in $L^1(Q)$ weak. Therefore, $f_0 \in L^1 \cap L^\infty(Q)$.

From the standard theory of existence of weak solutions for the Vlasov–Poisson system,[4,12,22,23] we obtain for all $\varepsilon > 0$

$$\|f^\varepsilon(t)\|_{L^p(Q)} \le \|f_0^\varepsilon\|_{L^p(Q)} \le C_0 < \infty, \quad 1 \le p \le \infty. \tag{27}$$

Then, by weak compactness arguments, there exists a function $f \in L^\infty(\mathbb{R}^+; L^1 \cap L^\infty(Q))$ such that f^ε (up to a subsequence) converges in $L^\infty(\mathbb{R}^+; L^\infty(Q))$ weak$-*$ to f and $⨍dx\, f^\varepsilon$ (up to a subsequence) converges in $L^\infty(\mathbb{R}^+; L^\infty(\mathbb{R}^d))$ weak$-*$ to $⨍dx\, f$. From properties of weak solutions for the Vlasov–Poisson system, weak solutions of (17) and (18) satisfy the following *a priori* bound:

$$\mathcal{E}(t) \le \mathcal{E}(0), \quad \forall t \ge 0, \tag{28}$$

where the total energy $\mathcal{E}(t)$ is defined by

$$\mathcal{E}(t) := \frac{1}{2}\int_Q dxdv\, |v|^2 f^\varepsilon(t,x,v) + \frac{\varepsilon}{2}\int_{\mathbb{T}^d} dx\, |E^\varepsilon(t,x)|^2.$$

From (28) and the initial data assumptions of Theorem 1, we infer that there exists a constant K_0, depending on C_0 such that

$$\int_Q dxdv\, |v|^2 f^\varepsilon(t,x,v) \le K_0, \quad \text{and} \quad \|E^\varepsilon\|_{L^\infty(\mathbb{R}^+;L^2(\mathbb{T}^d))} \le \frac{K_0}{\sqrt{\varepsilon}}. \tag{29}$$

Taking $\varphi \in \mathcal{D}(\mathbb{R}^+ \times Q)$ as a test function and using the L^2-scalar product (19), the weak formulation of Vlasov equation (17) reads

$$\varepsilon^2\langle f^\varepsilon, \partial_t\varphi\rangle + \langle f^\varepsilon, v\cdot\nabla_x\varphi\rangle + \varepsilon\langle f^\varepsilon, E^\varepsilon\cdot\nabla_v\varphi\rangle = 0. \tag{30}$$

Using the L^∞-bound for f^ε and *a priori* estimate (29) for the electric field E^ε, we obtain

$$|\varepsilon^2\langle f^\varepsilon, \partial_t\varphi\rangle| \le \varepsilon^2(2\pi)^{-d}\|\partial_t\varphi\|_{L^1(\mathbb{R}^+\times Q)}\|f^\varepsilon\|_{L^\infty(\mathbb{R}^+\times Q)} \le C\varepsilon^2 \tag{31}$$

and

$$\begin{aligned}\varepsilon|\langle f^\varepsilon, E^\varepsilon\cdot\nabla_v\varphi\rangle| &\le \varepsilon\|E^\varepsilon\|_{L^\infty(\mathbb{R}^+;L^2(\mathbb{T}^d))}\|f^\varepsilon\|_{L^\infty(\mathbb{R}^+;L^2(Q))}\|\nabla_v\varphi\|_{L^1(\mathbb{R}^+;L^2(\mathbb{R}^d;L^\infty(\mathbb{T}^d))}\\ &\le C\sqrt{\varepsilon}.\end{aligned} \tag{32}$$

Using (31) and (32) to pass to the limit $\varepsilon \to 0$ in (30), we obtain

$$v\cdot\nabla_x f = 0 \quad in \ \mathcal{D}'(\mathbb{R}^+ \times Q).$$

From Lemma 1, we infer that f is independent of x and $⨍dx\, f(t) = f(t)$ for a.e. $t > 0$. This proves point (i). For proving point (ii), we first define the charge density ρ^ε by

$$\rho^\varepsilon = \int_{\mathbb{R}^d} dv\, f^\varepsilon.$$

From a standard interpolation inequality (see, e.g., Refs. 12, 22, and 23), there exists a constant C_d depending on d such that

$$\|\rho^\varepsilon\|_{L^\infty(\mathbb{R}^+;L^{1+2/d}(\mathbb{T}^d))} \le C_d \|f^\varepsilon\|^{2/(2+d)}_{L^\infty(\mathbb{R}^+\times Q)} \|f^\varepsilon |v|^2\|^{d/(d+2)}_{L^\infty(\mathbb{R}^+;L^1(Q))} \le \kappa_0 < \infty, \tag{33}$$

where the constant κ_0 depends on C_0 but is independent of ε. From the Poisson equation (18), the bound (33) on the charge density ρ^ε and standard elliptic regularity estimates, we obtain

$$\|E^\varepsilon\|_{L^\infty(\mathbb{R}^+;W^{1,1+2/d}(\mathbb{T}^d))} \le c_0 < \infty, \tag{34}$$

where the constant c_0 depends on initial data but is independent of ε. Then, by weak compactness, there exists a function $E \in L^\infty(\mathbb{R}^+; W^{1,1+2/d}(\mathbb{T}^d))$ such that E^ε (up to a subsequence) converges in $L^\infty(\mathbb{R}^+; W^{1,1+2/d}(\mathbb{T}^d))$ weak$-\ast$ to E. To determine the limit point E, we use the Poisson equation (18). Observing that

$$\int_{\mathbb{R}^d} dv\, f = 1$$

and passing to the limit $\varepsilon \to 0$ in the Poisson equation (18), we obtain

$$\Delta\Phi = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^+ \times \mathbb{T}^d),$$

which leads to $\Phi = 0$ and $E = 0$ in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{T}^d)$. This ends the proof of point (ii). For point (iii), using the L^2-scalar product (19), we first write the following weak formulation of the Vlasov equation (17) being previously averaged in space:

$$\langle f^\varepsilon, \partial_t\varphi\rangle + \left\langle \frac{E^\varepsilon f^\varepsilon}{\varepsilon}, \nabla_v\varphi \right\rangle = 0, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^d). \tag{35}$$

Using *a priori* estimates (27) or point (i) of Theorem 1, we obtain, from (35),

$$\left| \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv\, \varphi \nabla_v \cdot \fint dx\, \frac{E^\varepsilon f^\varepsilon}{\varepsilon} \right| \le \|f^\varepsilon\|_{L^\infty([0,T]\times Q)} \|\partial_t \varphi\|_{L^1(\mathbb{R}^+\times\mathbb{R}^d)} \le C < \infty,$$

where C is independent of ε. This implies that $\nabla_v \cdot \fint dx E^\varepsilon f^\varepsilon/\varepsilon$ (up to a subsequence) converges in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d)$. Then, using point (i) of Theorem 1, we can pass to the limit $\varepsilon \to 0$ in (35) to obtain Eq. (21). For proving point (iv), we start by establishing some strong convergence properties for the sequences E^ε and Φ^ε. Using $\|E^\varepsilon\|_{W^s_{\text{loc}}(\mathbb{R}^+;L^1(\mathbb{T}^d))} \le \kappa$, with $s > 0$ [assumption (23) of Theorem 1], and $E^\varepsilon \in L^\infty(\mathbb{R}^+; W^{1,1+2/d}(\mathbb{T}^d))$, we obtain, from a Lions–Aubin theorem,[55] that the sequence

$$E^\varepsilon \ \text{ is compact in } \ L^{1+2/d}([0,T] \times \mathbb{T}^d) \ \text{ strong}, \quad \forall T > 0. \tag{36}$$

We next deal with the sequence Φ^ε. Using the Poisson equation (18), the bound (33) on the charge density ρ^ε, and standard elliptic regularity estimates, we obtain that $\Phi^\varepsilon \in L^\infty(\mathbb{R}^+; W^{2,1+2/d}(\mathbb{T}^d))$. Using the Sobolev embedding $W^{s,p}(\mathbb{R}^d) \hookrightarrow W^{r,q}(\mathbb{R}^d)$, with $s > r$, $d > (s-r)p$, and $p \le q \le dp/(d-(s-r)p)$, we obtain

$$L^\infty(\mathbb{R}^+; W^{2,1+2/d}(\mathbb{T}^d)) \hookrightarrow L^\infty(\mathbb{R}^+; W^{\delta,1+d/2}(\mathbb{T}^d)), \tag{37}$$

with $\max\{0, 2 - d/(1+2/d)\} < \delta < \max\{2, (-d^2+4d+4)/(d+2)\}$ and $d \le 4$. Using the embedding (37) and the bound $\|\partial_t\Phi^\varepsilon\|_{L^1_{\text{loc}}(\mathbb{R}^+;L^1(\mathbb{T}^d))} \le \kappa$ [assumption (23) of Theorem 1], we obtain from a Lions–Aubin theorem[55] that the sequence

$$\Phi^\varepsilon \ \text{ is compact in } \ L^{1+d/2}([0,T] \times \mathbb{T}^d) \ \text{ strong}, \quad \forall T > 0. \tag{38}$$

Multiplying the Vlasov equation (17) by $\varepsilon\Phi^\varepsilon$, then averaging in space, multiplying the result by a test function $\varphi \in \mathcal{D}([0,T] \times \mathbb{R}^d)$, and finally integrating with respect to the time and velocity variables, we obtain

$$\varepsilon \int_0^T dt \int_{\mathbb{R}^d} dv \fint dx\, \varphi \Phi^\varepsilon \partial_t f^\varepsilon + \frac{1}{\varepsilon}\int_0^T dt \int_{\mathbb{R}^d} dv \fint dx\, \varphi\Phi^\varepsilon v \cdot \nabla_x f^\varepsilon + \int_0^T dt \int_{\mathbb{R}^d} dv \fint dx\, \varphi \Phi^\varepsilon \nabla_v \cdot (E^\varepsilon f^\varepsilon) = 0. \tag{39}$$

Using integration by parts, we obtain from (39),

$$\varepsilon \int_0^T dt \int_{\mathbb{R}^d} dv \fint dx\, f^\varepsilon (\Phi^\varepsilon \partial_t \varphi + \varphi \partial_t \Phi^\varepsilon) - \int_0^T dt \int_{\mathbb{R}^d} dv\, \varphi v \cdot \fint dx \frac{E^\varepsilon f^\varepsilon}{\varepsilon} + \int_0^T dt \int_{\mathbb{R}^d} dv \fint dx\, f^\varepsilon \Phi^\varepsilon E^\varepsilon \cdot \nabla_v \varphi = 0. \tag{40}$$

Using the L^∞-bound (27) for f^ε and assumption (23), we obtain, for the first term of (40),

$$\varepsilon \left| \int_0^T dt \int_{\mathbb{R}^d} dv \fint dx\, f^\varepsilon (\Phi^\varepsilon \partial_t \varphi + \varphi \partial_t \Phi^\varepsilon) \right| \le \varepsilon (2\pi)^{-d} \| f^\varepsilon \|_{L^\infty([0,T]\times Q)} \left(\|\partial_t \varphi\|_{L^\infty(0,T;L^1(\mathbb{R}^d))} + \|\varphi\|_{L^\infty(0,T;L^1(\mathbb{R}^d))} \right) \left(\|\partial_t \Phi^\varepsilon\|_{L^\infty(0,T;L^1(\mathbb{T}^d))} + \|\Phi^\varepsilon\|_{L^\infty(0,T;L^1(\mathbb{T}^d))} \right) \le C\varepsilon. \tag{41}$$

Using (36) and (38), we infer that the product $\Phi^\varepsilon E^\varepsilon$ converges strongly in $L^1([0,T]\times \mathbb{T}^d)$. Using this strong convergence, the weak convergence of f^ε in $L^\infty([0,T]\times Q)$ weak–$*$ and the fact that the limit point of E^ε vanishes, we obtain for the third term of (40),

$$\left| \int_0^T dt \int_{\mathbb{R}^d} dv \fint dx\, f^\varepsilon \Phi^\varepsilon E^\varepsilon \cdot \nabla_v \varphi \right| \to 0 \text{ as } \varepsilon \to 0. \tag{42}$$

Using (40)–(42), we obtain

$$\int_0^T dt \int_{\mathbb{R}^d} dv\, \varphi v \cdot \fint dx \frac{E^\varepsilon f^\varepsilon}{\varepsilon} \to 0 \text{ as } \varepsilon \to 0. \tag{43}$$

Choosing a test function φ such that $0_{\mathbb{R}^d_v} \notin \operatorname{supp}(\varphi)$ and passing to the limit $\varepsilon \to 0$ in (35), we then obtain from (43),

$$\int_0^T dt \int_{\mathbb{R}^d} dv\, f \partial_t \varphi = 0. \tag{44}$$

Since $f \in L^\infty([0,T]\times \mathbb{R}^d)$, Eq. (44) is valid for any $\varphi \in \mathcal{D}([0,T]\times\mathbb{R}^d)$; hence, we obtain (24)–(26), which ends the proof. □

Few remarks on Theorem 1 are in order.

Remark 2 (electric energy). A priori estimate (29) for E^ε is not optimal. In fact, we have $E^\varepsilon \in L^\infty(\mathbb{R}^+; L^2(\mathbb{T}^d))$ for $d \le 4$, uniformly with respect to ε. Indeed, for $d \le 2$, it is obvious from estimate (34). For $3 \le d \le 4$, it comes from estimate (34) and the Sobolev embedding $W^{1,1+2/d}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$, with $q = d(d+2)/(d-2)/(d+1)$ and $d > 2$.

Remark 3 (time compactness).

1. *To obtain time compactness, there are a priori three ways. The first one is to obtain time compactness for the electric field by using standard control on the charge current j^ε (see, e.g., Refs. 12, 22, and 23), and the Ampère equation given by $\varepsilon^2 \partial_t E^\varepsilon + j^\varepsilon = 0$. This method fails because the presence of the factor ε^2 in front the time partial derivative does not give uniform bound (with respect to ε) for $\partial_t E^\varepsilon$. We obtain the same result from the charge conservation law (to obtain time compactness on the charge density ρ^ε and thus on E^ε via the Poisson equation), since the latter can be recovered by applying the spatial divergence operator to the Ampère equation. The second method is to use averaging lemmas.[24] With this method, we only obtain compactness in the space variables but not in the time variable because in the limit $\varepsilon \to 0$, the term $\varepsilon^2 \partial_t f^\varepsilon$ in the Vlasov equation disappears.[24,53] A third way is to obtain time compactness for the distribution function instead of the electric field. For this, we can show uniform convergence with respect to time in a weak topology for the phase-space variables. Showing time equi-continuity for the distribution function requires using the Vlasov equation. Here again, the presence of the factor ε^2 in front of the term $\partial_t f^\varepsilon$ in the Vlasov equation makes this method to fail.*
2. *Point (iv) of Theorem 1 shows that the lack of time compactness is, in fact, a necessary condition for obtaining a genuine or a non-degenerate diffusion equation in the limit $\varepsilon \to 0$.*

Remark 4 (boundary conditions). As shown in the Proof of Theorem 1, the relation $v \cdot \nabla_x f = 0$ is a direct consequence of a priori estimates. By a direct computation using Fourier series (see the Proof of Lemma 1), it has been proved that the limit point f is independent of the space variable x. This is the ergodic property of the torus. Obviously, the same property is true when the torus is replaced by any domain where the free flow trajectory $(x_0, v_0) \mapsto (x_0 + v_0 t, v_0)$ with specular reflections at the boundary are dense (this is a definition of ergodicity). Extending the present analysis to this more general case may be very useful.

Remark 5 ("cheap" Landau damping).

1. *What we proved for the rescaled Vlasov–Poisson system, given by (17)* and *(18), is a "cheap" version of the Mouhot–Villani version of the Landau damping,*[48] *i.e. that (under convenient hypotheses of regularity for initial conditions and smallness for initial perturbations) the self-consistent electric field E of the Vlasov–Poisson system* (7) and (8) *vanishes strongly when $t \to +\infty$, while the distribution f relaxes, in a weak topology, toward a spatially homogeneous stationary solution of the Vlasov–Poisson system. Indeed, if ε is the ratio of the electric field $E(t)$ of* (7) and (8) *to the electric field $E^\varepsilon(t)$ of* (17) and (18) *[implying that $|E^\varepsilon(t)| > |E(t)|$], with the change of time scale $t \to t/\varepsilon^2$, the Vlasov–Poisson system [*(7) and (8)*] becomes the rescaled Vlasov–Poisson system [*(17) and (18)*]. In other words, the limit $t \to +\infty$ in* (7) and (8) *is equivalent to the limit $\varepsilon \to 0$ in* (17) and (18)*, and $\varepsilon = 1/\sqrt{t}$ stands for the smallest rate at which the electric field $E(t)$ tends to zero when $t \to +\infty$.*
2. *By considering the rescaled Vlasov–Poisson system [(17)–(18)*] *in the framework of the Landau damping, we observe that under the hypotheses of Ref. 48 the electric field converges strongly to zero. From Theorem 1, this strong convergence corresponds to a zero diffusion. The Mouhot–Villani result*[48] *is obtained for small perturbations (in some analytic norms) of a stable equilibrium profile (in velocity variables). Here, we are interested in unstable equilibrium profiles that lead to a non-zero diffusion in the velocity space.*

The velocity diffusion operator should arise when we pass to the limit in the term $\varepsilon^{-1}\nabla_v \cdot \fint dx\, E^\varepsilon f^\varepsilon$. A rigorous proof of this fact remains an open issue and will be the matter of a future work. Nevertheless, we can show, at least formally, what is the structure of this term by using a simple iteration of the Duhamel formula. This is the aim of Sec. III B.

B. Duhamel formula and Fick-type law

Here, we derive formally a Fick-type law for the flux term

$$\overline{\fint dx\, \frac{E^\varepsilon f^\varepsilon}{\varepsilon}},$$

appearing in (21). Most of developments of this section are formal, but they allow us to point out the difficulties for showing rigorously the diffusion limit. This Fick-type law can be obtained from two ways. The first way is a global-in-time approach, which involves the initial condition f_0^ε, while the second one, a local-in-time approach, does not. Each approach has its advantages (Lemmas 2 and 3) and drawbacks (Remarks 7 and 10). In addition, for both approaches, the absence of time decorrelation properties prevents us to determine the structure and the properties of the diffusion matrix. Nevertheless, a formal WKB approximation allows us to obtain a non-negative diffusion matrix in the non-self-consistent case.

1. Global-in-time approach

Using the Duhamel formula and (20), we obtain from the Vlasov equation (17) the following representation formula for $f^\varepsilon(t)$, solution to (17) and (18):

$$f^\varepsilon(t) = S_t^\varepsilon f_0^\varepsilon - \frac{1}{\varepsilon}\int_0^t ds\, S_{t-s}^\varepsilon E^\varepsilon(s)\cdot\nabla_v f^\varepsilon(s). \tag{45}$$

Substituting (45) into

$$-\int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv\, \varphi \nabla_v \cdot \fint dx\, \frac{E^\varepsilon f^\varepsilon}{\varepsilon} = \frac{1}{\varepsilon}\int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \fint dx\, \nabla_v\varphi \cdot E^\varepsilon f^\varepsilon, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^d),$$

we obtain

$$-\int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv\, \varphi \nabla_v \cdot \fint dx\, \frac{E^\varepsilon f^\varepsilon}{\varepsilon} = T_1^\varepsilon(\varphi) + T_2^\varepsilon(\varphi), \tag{46}$$

where

$$T_1^\varepsilon(\varphi) := \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \, \frac{1}{\varepsilon} \nabla_v \varphi(t,v) \cdot \fint dx \, E^\varepsilon(t,x) f_0^\varepsilon(x - vt/\varepsilon^2, v)$$

and

$$T_2^\varepsilon(\varphi) := -\int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \, \frac{1}{\varepsilon^2} \nabla_v \varphi(t,v) \cdot \int_0^t ds \fint dx \, E^\varepsilon(t,x) E^\varepsilon(s, x - v(t-s)/\varepsilon^2) \cdot (\nabla_v f^\varepsilon)(s, x - v(t-s)/\varepsilon^2, v).$$

For the term T_1^ε, *we have the following lemma*:

Lemma 2. Assume that f_0^ε satisfies the hypotheses of Theorem 1. *In addition, we suppose that there exists a constant $\mathcal{C}_0$, independent of ε, such that for $|\alpha| \le 1$,*

$$\sum_{k \in \mathbb{Z}_*^d} \left(|k|^{-1} \| \partial_v^\alpha \hat{f}_0^\varepsilon(k) \|_{L^1(\mathbb{R}^d)} \right)^2 \le \mathcal{C}_0 \qquad \text{if } d = 1 \text{ and}$$
$$\sum_{k \in \mathbb{Z}_*^d} \left(|k|^{-2} \| \partial_v^\alpha \hat{f}_0^\varepsilon(k) \|_{L^1(\mathbb{R}^d)} \right)^{1+2/d} \le \mathcal{C}_0 \qquad \text{if } d \ge 2. \tag{47}$$

Then,

$$T_1^\varepsilon \rightharpoonup 0 \ \ in \ \ \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d). \tag{48}$$

Proof. Using Fourier series and the zero-mean electrostatic condition (10), we rewrite the term T_1^ε as

$$T_1^\varepsilon(\varphi) = \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \sum_{k \in \mathbb{Z}_*^d} \frac{1}{\varepsilon} \nabla_v \varphi(t,v) \cdot \hat{E}^\varepsilon(t,-k) \hat{f}_0^\varepsilon(k,v) \exp(-ik \cdot vt/\varepsilon^2).$$

Using a velocity integration by parts, we obtain

$$T_1^\varepsilon(\varphi) = -i\varepsilon \int_{\mathbb{R}^+} dt \, \frac{1}{t} \int_{\mathbb{R}^d} dv \sum_{k \in \mathbb{Z}_*^d} \nabla_v \cdot \left(\frac{k}{|k|^2} \nabla_v \varphi(t,v) \cdot \hat{E}^\varepsilon(t,-k) \hat{f}_0^\varepsilon(k,v) \right) \exp(-ik \cdot vt/\varepsilon^2),$$

which leads to

$$|T_1^\varepsilon(\varphi)| \le \varepsilon \int_{\mathbb{R}^+} dt \, \frac{1}{t} \sum_{k \in \mathbb{Z}_*^d} |\hat{E}^\varepsilon(t,k)| |k|^{-1} \left(\|\nabla_v^2 \varphi(t)\|_{L^\infty(\mathbb{R}^d)} \|\hat{f}_0^\varepsilon(k)\|_{L^1(\mathbb{R}^d)} + \|\nabla_v \varphi(t)\|_{L^\infty(\mathbb{R}^d)} \|\nabla_v \hat{f}_0^\varepsilon(k)\|_{L^1(\mathbb{R}^d)} \right). \tag{49}$$

Using the bound (34) and the Hausdorff–Young inequality, we obtain for $d \ge 2$,

$$\| |k| \hat{E}^\varepsilon \|_{L^\infty(\mathbb{R}^+; \ell^{1+d/2}(\mathbb{Z}^d))} \le \| E^\varepsilon \|_{L^\infty(\mathbb{R}^+; W^{1,1+2/d}(\mathbb{T}^d))} \le c_0. \tag{50}$$

Using Hölder's inequality, (50), and assumption (47), we obtain from (49),

$$|T_1^\varepsilon(\varphi)| \le \varepsilon 2 c_0 \mathcal{C}_0^{d/(d+2)} \|\varphi/t\|_{L^1(\mathbb{R}^+; W^{2,\infty}(\mathbb{R}^d))}.$$

In the same way, using Remark 2 and the Cauchy–Scharwz inequality, we obtain for $d = 1$,

$$|T_1^\varepsilon(\varphi)| \le \varepsilon 2 \mathcal{C}_0^{1/2} \|E^\varepsilon\|_{L^\infty(\mathbb{R}^+; L^2(\mathbb{T}^d))} \|\varphi/t\|_{L^1(\mathbb{R}^+; W^{2,\infty}(\mathbb{R}^d))},$$

which ends the Proof of Lemma 2. □

Remark 6. In Lemma 2, the regularity assumption for f_0^ε might be refined, but with the presence of the factor ε^{-1} in the term T_1^ε, some mixing-type hypotheses seem compulsory.

We now deal with the term T_2^ε. Performing the change of time variable $s = t - \varepsilon^2\sigma$, followed by the change of space variable $x' = x - \sigma v$, and using a velocity integration by parts and x-periodicity, we obtain

$$T_2^\varepsilon(\varphi) = \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \fint dx \int_0^{t/\varepsilon^2} d\sigma \\ f^\varepsilon(t - \varepsilon^2\sigma, x, v)\nabla_v \cdot \big(E^\varepsilon(t - \varepsilon^2\sigma, x) \otimes E^\varepsilon(t, x + v\sigma)\nabla_v\varphi(t, v)\big). \tag{51}$$

Using the time characteristic function $\chi_{[0,t/\varepsilon^2]}(\sigma)$, Eq. (51) can be recast as

$$T_2^\varepsilon(\varphi) = J^\varepsilon(\varphi) + M^\varepsilon(\varphi), \tag{52}$$

where

$$J^\varepsilon(\varphi) = \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv\, f(t,v)\nabla_v \cdot \Big(\int_{\mathbb{R}^+} d\sigma \fint dx\, \chi_{[0,t/\varepsilon^2]}(\sigma) E^\varepsilon(t - \varepsilon^2\sigma, x) \otimes E^\varepsilon(t, x + v\sigma)\, \nabla_v\varphi(t,v)\Big)$$

and

$$M^\varepsilon(\varphi) := \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \int_{\mathbb{R}^+} d\sigma \\ \fint dx\, \chi_{[0,t/\varepsilon^2]}(\sigma)\big(f^\varepsilon(t - \varepsilon^2\sigma, x, v) - f(t,v)\big)\nabla_v \cdot \big(E^\varepsilon(t - \varepsilon^2\sigma, x) \otimes E^\varepsilon(t, x + v\sigma)\nabla_v\varphi(t,v)\big).$$

If we assume that

$$\lim_{\varepsilon\to 0} J^\varepsilon(\varphi) \quad \text{exists} \tag{53}$$

and

$$\lim_{\varepsilon\to 0} M^\varepsilon(\varphi) = 0, \tag{54}$$

then we obtain from (52),

$$\lim_{\varepsilon\to 0} T_2^\varepsilon(\varphi) = \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv\, f(t,v)\nabla_v \cdot \Big(\mathscr{D}(t,v)^T \nabla_v\varphi(t,v)\Big), \tag{55}$$

with

$$\mathscr{D}(t,v) := \lim_{\varepsilon\to 0} \int_{\mathbb{R}^+} d\sigma \fint dx\, \chi_{[0,t/\varepsilon^2]}(\sigma) E^\varepsilon(t, x + v\sigma) \otimes E^\varepsilon(t - \varepsilon^2\sigma, x) \\ = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^+} d\sigma \fint dx\, \chi_{[0,t/\varepsilon^2]}(\sigma) E^\varepsilon(t, x) \otimes E^\varepsilon(t - \varepsilon^2\sigma, x - v\sigma). \tag{56}$$

Using (46), (48), and (55) to pass to the limit $\varepsilon \to 0$ in (35), we obtain the following diffusion equation:

$$\partial_t f(t,v) - \nabla_v \cdot (\mathscr{D}(t,v)\nabla_v f(t,v)) = 0 \quad in\ \ \mathcal{D}'([0,T] \times \mathbb{R}^d). \tag{57}$$

Few remarks are now in order.

Remark 7 (open issues).

1. *All computations involving the term T_2^ε are formal and must be justified in a convenient functional framework. In order to justify (53), we have to show that*

$$R^\varepsilon(t, \sigma, x, v) := \chi_{[0,t/\varepsilon^2]}(\sigma)\nabla_v \cdot \big(E^\varepsilon(t - \varepsilon^2\sigma, x) \otimes E^\varepsilon(t, x + v\sigma)\nabla_v\varphi(t,v)\big)$$

   converges weakly in $L^1(\mathbb{R}_t^+ \times \mathbb{R}_\sigma^+ \times Q)$. In order to prove (54) and justify (55), we have to show that R^ε converges strongly in $L^1(\mathbb{R}_t^+ \times \mathbb{R}_\sigma^+ \times Q)$, since $f^\varepsilon(t - \varepsilon^2\sigma, x, v) - f(t,v) \rightharpoonup 0$ in $L^\infty(\mathbb{R}_t^+ \times \mathbb{R}_\sigma^+ \times Q)$ weak$-*$. We observe that a crucial point is to obtain enough integrability with respect the time variable σ, uniformly in ε.

2. *As already observed, the bound* $\|E^\varepsilon\|_{L^\infty(\mathbb{R}^+;W^{1,1+2/d}(\mathbb{T}^d))} \le c_0 < \infty$ *does not imply strong convergence (because of the lack of time control or compactness) for the electric field* E^ε*, which would help to justify the above formal computations for the term* T_2^ε*. However, this lack of time compactness is, in fact, necessary if we do not want to obtain a trivial equation, as stated in point (iv) of Theorem 1. Indeed, from point (iv) of Theorem 1, time compactness entails a strong convergence to zero of the electric field* E^ε*. This implies the vanishing of the diffusion matrix* $\mathscr{D}$ *given by (56). Without time compactness, the electric field* E^ε *always converges weakly to zero, but not the quadratic electric tensor* $E^\varepsilon \otimes E^\varepsilon$ *(this is a property of weak convergence), which implies a non-trivial diffusion matrix* $\mathscr{D}$*. Therefore, weak convergence seems mandatory to obtain a diffusion limit.*
3. *Instead of time compactness, time decorrelation properties could help to justified rigorously above computations. In the presence of a non-self-consistent but stochastic electric field, with convenient hypotheses, some time decorrelation properties allow us to justify rigorously the limit of the Vlasov equation (17) toward diffusion equations* (56) and (57). *This is the object of Sec. IV.*

Remark 8 (periodic or quasi-periodic time oscillations). Since the defect of time compactness means that the system contains fast oscillations in time, it would be tempting to apply the analysis of this section to the case of a non-self-consistent deterministic electric field (satisfying convenient regularity assumptions) with two time scales, one being slow and not periodic and the other being fast and (quasi-)periodic. Such a standard homogenization problem would lead to solve a hierarchy of equations where the free-streaming operator $v \cdot \nabla_x$*, with periodic boundary condition, must be inverted at each stage of the hierarchy. Since the free-flow operator is not a Fredholm operator, the Fredholm alternative does not hold for such a transport equation. On the contrary, when considering a non-self-consistent stochastic electric field, the situation is completely different and we can obtain a diffusion behavior for the statistical average of the distribution function. This is what is done in Sec. IV.*

An explicit form of the diffusion matrix in the non-self-consistent deterministic case.

Here, we pursue a little bit further the above formal analysis to explicit the structure of the diffusion matrix (56) by constructing a well-suited non-self-consistent deterministic electric field. This gives an example of diffusion matrix (56), which is not zero and non-negative. Introducing the Fourier series decomposition of E^ε,

$$E^\varepsilon(t,x) = \sum_{k\in\mathbb{Z}^d} e^{\mathrm{i}k\cdot x}\hat{E}^\varepsilon(t,k),$$

we assume the formal WKB expansion for the Fourier mode $\hat{E}^\varepsilon(t,k)$,

$$\hat{E}^\varepsilon(t,k) = \sum_{j\ge 0}\varepsilon^j \hat{E}_j(t,k,\Omega(t,k)/\varepsilon^2), \tag{58}$$

where complex vector-valued functions $(k,\tau)\mapsto \hat{E}_j(t,k,\tau)$ are 2π-periodic with respect to the variable τ. Here, functions $\hat{E}_j(t,k,\Omega(t,k)/\varepsilon^2)$ are Hermitian, i.e.,

$$\hat{E}_j^*(t,k,\Omega(t,k)/\varepsilon^2) = \hat{E}_j(t,-k,\Omega(t,-k)/\varepsilon^2),$$

and the real-valued function $k\mapsto\Omega(t,k)$ is odd with respect to the variable k. As a first approximation of (58), we obtain

$$\hat{E}^\varepsilon(t,k) = \hat{E}_0(t,k)\exp\left(-\mathrm{i}\frac{\Omega(t,k)}{\varepsilon^2}\right) + \mathcal{O}(\varepsilon), \tag{59}$$

where the real vector-valued function $k\mapsto \hat{E}_0(t,k)$ is even with respect to the variable k. Using (59) and time Taylor expansions, we obtain from the definition of the diffusion matrix (56),

$$\begin{aligned}\mathscr{D}(t,v) &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^+} d\sigma \sum_{k\in\mathbb{Z}^d}\chi_{[0,t/\varepsilon^2]}(\sigma)\exp(\mathrm{i}k\cdot v\sigma)\hat{E}^\varepsilon(t,k)\otimes\hat{E}^\varepsilon(t-\varepsilon^2\sigma,-k)\\ &= \lim_{\varepsilon\to 0}\Bigg(\int_{\mathbb{R}^+} d\sigma\sum_{k\in\mathbb{Z}^d}\chi_{[0,t/\varepsilon^2]}(\sigma)\exp\big(\mathrm{i}[k\cdot v\sigma + \Omega(t-\varepsilon^2\sigma,k)/\varepsilon^2 - \Omega(t,k)/\varepsilon^2]\big)\\ &\quad \hat{E}_0(t,k)\otimes\hat{E}_0(t-\varepsilon^2\sigma,-k) + \mathcal{O}(\varepsilon)\Bigg)\\ &= \lim_{\varepsilon\to 0}\Bigg(\int_{\mathbb{R}^+} d\sigma\sum_{k\in\mathbb{Z}^d}\chi_{[0,t/\varepsilon^2]}(\sigma)\exp(-\mathrm{i}\sigma[\partial_t\Omega(t,k) - k\cdot v])\hat{E}_0(t,k)\otimes\hat{E}_0(t,k) + \mathcal{O}(\varepsilon)\Bigg).\end{aligned} \tag{60}$$

Using

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^+} d\sigma\,\chi_{[0,t/\varepsilon^2]}(\sigma)e^{-\mathrm{i}\sigma\tau} = \pi\delta(\tau) - \mathrm{i}\,\mathrm{p.v.}\left(\frac{1}{\tau}\right)\ \ in\ \mathcal{D}'(\mathbb{R})$$

and parity of functions $\Omega(t,k)$ and $\hat{E}_0(t,k)$, we obtain from (60),

$$\mathscr{D}(t,v) = \pi\sum_{k\in\mathbb{Z}^d}\hat{E}_0(t,k)\otimes\hat{E}_0(t,k)\delta(\partial_t\Omega(t,k) - k\cdot v). \tag{61}$$

If we assume $\Omega(t,k) = \omega(k)t$, then (61) is the diffusion matrix given by the quasilinear theory,[21,39] i.e.,

$$\mathscr{D}(t,v) = \pi\sum_{k\in\mathbb{Z}^d}\hat{E}_0(t,k)\otimes\hat{E}_0(t,k)\delta(\omega(k) - k\cdot v). \tag{62}$$

Remark 9. If the electric field E^ε derives from a potential Φ^ε, i.e., $E^\varepsilon(t,x) = -\nabla\Phi^\varepsilon(t,x)$, then following the above computations, a WKB expansion of $\Phi^\varepsilon(t,x)$ similar to (58) leads to the diffusion matrix

$$\mathscr{D}(t,v) = \pi\sum_{k\in\mathbb{Z}^d}|\hat{E}_0(t,k)|^2\,\frac{k\otimes k}{|k|^2}\,\delta(\omega(k) - k\cdot v).$$

2. Local-in-time approach

We first integrate, with respect to the time variable, the space-averaged Vlasov equation

$$\partial_t ⨍ dx\, f^\varepsilon + \nabla_v\cdot ⨍ dx\,\frac{E^\varepsilon f^\varepsilon}{\varepsilon} = 0$$

between the time t and $t+\theta$, with $\theta > 0$. Then, we multiply the result by $\varphi\in\mathcal{D}(\mathbb{R}^+\times\mathbb{R}^d)$ and we perform an integration with respect to the time and velocity variables. Finally, using the L^2-scalar product (19) and a velocity integration by parts, we obtain

$$\left\langle\frac{f^\varepsilon(t+\theta) - f^\varepsilon(t)}{\theta},\varphi\right\rangle = \frac{1}{\varepsilon\theta}\int_{\mathbb{R}^+}dt\int_{\mathbb{R}^d}dv\int_t^{t+\theta}ds ⨍ dx\, f^\varepsilon(s)E^\varepsilon(s)\cdot\nabla_v\varphi(t,v). \tag{63}$$

Using the Duhamel formula and the notation (20), we obtain from the Vlasov equation (17) the following representation formula for $f^\varepsilon(s)$:

$$f^\varepsilon(s) = S^\varepsilon_{s-t+\hat{\theta}}f^\varepsilon(t-\hat{\theta}) - \frac{1}{\varepsilon}\int_{t-\hat{\theta}}^{s}d\sigma\, S^\varepsilon_{s-\sigma}E^\varepsilon(\sigma)\cdot\nabla_v f^\varepsilon(\sigma), \tag{64}$$

with $\hat{\theta}$ being an arbitrary non-negative time. Substituting (64) into (63), we obtain

$$\left\langle\frac{f^\varepsilon(t+\theta) - f^\varepsilon(t)}{\theta},\varphi\right\rangle = \mathcal{T}_1^\varepsilon(\varphi) + \mathcal{T}_2^\varepsilon(\varphi), \tag{65}$$

where

$$\mathcal{T}_1^\varepsilon(\varphi) := \int_{\mathbb{R}^+}dt\int_{\mathbb{R}^d}dv\int_t^{t+\theta}ds ⨍ dx\,\frac{1}{\varepsilon\theta}E^\varepsilon(s)\cdot\nabla_v\varphi(t,v)S^\varepsilon_{s-t+\hat{\theta}}f^\varepsilon(t-\hat{\theta}) \tag{66}$$

and

$$\mathcal{T}_2^\varepsilon(\varphi) := \int_{\mathbb{R}^+}dt\int_{\mathbb{R}^d}dv \int_t^{t+\theta}ds\int_{t-\hat{\theta}}^{s}d\sigma ⨍ dx\,\frac{1}{\varepsilon^2\theta}S^\varepsilon_{s-\sigma}f^\varepsilon(\sigma)\nabla_v\cdot\left(S^\varepsilon_{s-\sigma}E^\varepsilon(\sigma,x)\otimes E^\varepsilon(s,x)\nabla_v\varphi(t,v)\right).$$

For the term $\mathcal{T}_1^\varepsilon$, we assume that

$$\mathcal{T}_1^\varepsilon \rightharpoonup 0 \ \ in \ \ \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d). \tag{67}$$

We now deal with the term $\mathcal{T}_2^\varepsilon$. Using x-periodicity and the change of time variable $s' = s - t$, the term $\mathcal{T}_2^\varepsilon$ becomes

$$\mathcal{T}_2^\varepsilon(\varphi) = \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \int_0^\theta ds \int_{t-\hat{\theta}}^{t+s} d\sigma ⨍ dx \\ \frac{1}{\varepsilon^2\theta} f^\varepsilon(\sigma, x, v) \nabla_v \cdot \big(E^\varepsilon(\sigma, x) \otimes E^\varepsilon(t+s, x+v(t+s-\sigma)/\varepsilon^2) \nabla_v \varphi(t,v) \big). \tag{68}$$

Using the change of time variable $\sigma' = (t+s-\sigma)/\varepsilon^2$, Eq. (68) becomes

$$\mathcal{T}_2^\varepsilon(\varphi) = \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \int_0^\theta ds \int_0^{(s+\hat{\theta})/\varepsilon^2} d\sigma ⨍ dx \\ \frac{1}{\theta} f^\varepsilon(t+s-\varepsilon^2\sigma, x, v) \nabla_v \cdot \big(E^\varepsilon(t+s-\varepsilon^2\sigma, x) \otimes E^\varepsilon(t+s, x+v\sigma) \nabla_v \varphi(t,v) \big). \tag{69}$$

Taking $\theta = \varepsilon^2 \bar{\tau}$ and $\hat{\theta} = \varepsilon^2 \eta$ and using the change of time variable $s' = s/\varepsilon^2$, followed by the change of time variable $\sigma' = \sigma - \eta$, Eq. (69) becomes

$$\mathcal{T}_2^\varepsilon(\varphi) = \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \, \frac{1}{\bar{\tau}} \int_0^{\bar{\tau}} ds \int_{-\eta}^{s} d\sigma ⨍ dx \, f^\varepsilon(t+(s-\eta-\sigma)\varepsilon^2, x, v) \\ \nabla_v \cdot \big(E^\varepsilon(t+(s-\eta-\sigma)\varepsilon^2, x) \otimes E^\varepsilon(t+s\varepsilon^2, x+v(\sigma+\eta)) \nabla_v \varphi(t,v) \big).$$

This equation can be recast as

$$\mathcal{T}_2^\varepsilon(\varphi) = \mathcal{J}^\varepsilon(\varphi) + \mathcal{M}^\varepsilon(\varphi), \tag{70}$$

where

$$\mathcal{J}^\varepsilon(\varphi) = \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv f(t,v) \nabla_v \cdot \bigg(\frac{1}{\bar{\tau}} \int_0^{\bar{\tau}} ds \int_{-\eta}^{s} d\sigma ⨍ dx \\ E^\varepsilon(t+(s-\eta-\sigma)\varepsilon^2, x) \otimes E^\varepsilon(t+s\varepsilon^2, x+v(\sigma+\eta)) \nabla_v \varphi(t,v) \bigg)$$

and

$$\mathcal{M}^\varepsilon(\varphi) := \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \, \frac{1}{\bar{\tau}} \int_0^{\bar{\tau}} ds \int_{-\eta}^{s} d\sigma ⨍ dx \big(f^\varepsilon(t+(s-\eta-\sigma)\varepsilon^2, x, v) - f(t,v) \big) \\ \nabla_v \cdot \big(E^\varepsilon(t+(s-\eta-\sigma)\varepsilon^2, x) \otimes E^\varepsilon(t+s\varepsilon^2, x+v(\sigma+\eta)) \nabla_v \varphi(t,v) \big).$$

The next lemma justifies that in the case where $\bar{\tau}$ and η are finite, the term $\mathcal{J}^\varepsilon(\varphi)$ has a limit as $\varepsilon \to 0$. Defining

$$\mathscr{D}^\varepsilon(t,v) := \frac{1}{\bar{\tau}} \int_0^{\bar{\tau}} ds \int_{-\eta}^{s} d\sigma ⨍ dx \, E^\varepsilon(t+s\varepsilon^2, x+v(\sigma+\eta)) \otimes E^\varepsilon(t+(s-\eta-\sigma)\varepsilon^2, x), \tag{71}$$

we have the following lemma:

Lemma 3. Let $\bar{\tau}$ and η be finite. Then, $\mathcal{J}^\varepsilon$ has a limit in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d)$ such that

$$\lim_{\varepsilon\to 0} \mathcal{J}^\varepsilon(\varphi) = \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \, f \nabla_v \cdot \big(\mathscr{D}^T \nabla_v \varphi \big), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^d), \tag{72}$$

where $\mathscr{D}$ is the weak limit of $\mathscr{D}^\varepsilon$ (up to a subsequence) in the following sense:

$$\mathscr{D}^\varepsilon \rightharpoonup \mathscr{D} \quad in \quad L^1_{\text{loc}}(\mathbb{R}^+; W^{1,1}_{\text{loc}}(\mathbb{R}^d)) \quad weak \tag{73}$$

and

$$\mathscr{D}^\varepsilon \rightharpoonup \mathscr{D} \quad in \quad L^\infty(\mathbb{R}^+; L^\infty(\mathbb{R}^d)) \quad weak-*. \tag{74}$$

Proof. Since $f \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$, to prove (72) and (73), it is sufficient to show that the term

$$g^\varepsilon(t,v) := \nabla_v \cdot \left(\mathscr{D}^\varepsilon(t,v)^T \nabla_v \varphi(t,v)\right)$$

converges weakly in $L^1(\mathbb{R}^+ \times \mathbb{R}^d)$ as $\varepsilon \to 0$, i.e., for the weak topology $\sigma(L^1, L^\infty)$. For this, we appeal to the Dunford–Pettis theorem. First we show that $g^\varepsilon \in L^1(\mathbb{R}^+ \times \mathbb{R}^d)$ uniformly in ε. For this, we use the fact that weak solutions of (17) and (18) are such that $E^\varepsilon \in L^\infty(\mathbb{R}^+; W^{1,1+2/d}(\mathbb{T}^d))$ uniformly with respect to ε. For $d = 2$, we use the L^2 duality and $E^\varepsilon \in L^\infty(\mathbb{R}^+; L^2(\mathbb{T}^d))$ uniformly in ε (see Remark 2). For $d = 3$, we use the L^p–L^q duality with $(p,q) = (1 + 2/d, 1 + d/2)$ and the Sobolev embedding $W^{1,1+2/d}(\mathbb{T}^d) \hookrightarrow L^{1+d/2}(\mathbb{T}^d)$. Since the case $d = 1$ is simpler [using the regularity $E^\varepsilon \in L^\infty(\mathbb{R}^+ \times \mathbb{T}^d)$ uniformly in ε], we only give the proof for $d \geq 2$. Then, for $d \geq 2$, using Hölder's inequality, we obtain

$$\begin{aligned}
\|g^\varepsilon\|_{L^1(\mathbb{R}^+\times\mathbb{R}^d)} \leq{}& \Bigg\| \frac{1}{\bar\tau}\sum_{i,j}\int_0^{\bar\tau} ds \int_{-\eta}^{s} d\sigma \fint dx\,(\sigma+\eta) \\
& E_i^\varepsilon(t+(s-\eta-\sigma)\varepsilon^2, x)\,(\partial_{x_j}E^\varepsilon)(t+s\varepsilon^2, x+v(\sigma+\eta))\partial_{v_j}\varphi(t,v)\Bigg\|_{L^1(\mathbb{R}^+\times\mathbb{R}^d)} \\
& + \Bigg\| \sum_{i,j}\frac{1}{\bar\tau}\int_0^{\bar\tau} ds \int_{-\eta}^{s} d\sigma \fint dx\,(\sigma+\eta) \\
& E_j^\varepsilon(t+(s-\eta-\sigma)\varepsilon^2, x)E_j^\varepsilon(t+s\varepsilon^2, x+v(\sigma+\eta))\partial^2_{v_iv_j}\varphi(t,v)\Bigg\|_{L^1(\mathbb{R}^+\times\mathbb{R}^d)} \\
\leq{}& (\bar\tau^2 + 3\bar\tau\eta + 3\eta^2)(2\pi)^{-d}\|E^\varepsilon\|_{L^\infty(\mathbb{R}^+;L^{1+d/2}(\mathbb{T}^d))} \\
& \left\{\|\nabla_x E^\varepsilon\|_{L^\infty(\mathbb{R}^+;L^{1+2/d}(\mathbb{T}^d))}\|\nabla_v\varphi\|_{L^1(\mathbb{R}^+\times\mathbb{R}^d)} + \|E^\varepsilon\|_{L^\infty(\mathbb{R}^+;L^{1+2/d}(\mathbb{T}^d))}\|\nabla_v^2\varphi\|_{L^1(\mathbb{R}^+\times\mathbb{R}^d)}\right\} \\
<{}& \infty.
\end{aligned}$$

For showing uniform equi-integrability of the family g^ε, we take $A \subset \mathbb{R}^+ \times \mathbb{R}^d$ such that $|A| \leq \delta$, with δ small. Following the above computations, we obtain

$$\begin{aligned}
\|g^\varepsilon\|_{L^1(A)} \leq{}& (\bar\tau^2 + 3\bar\tau\eta + 3\eta^2)(2\pi)^{-d}\|E^\varepsilon\|_{L^\infty(\mathbb{R}^+;L^{1+d/2}(\mathbb{T}^d))} \\
& \left\{\|\nabla_x E^\varepsilon\|_{L^\infty(\mathbb{R}^+;L^{1+2/d}(\mathbb{T}^d))}\|\nabla_v\varphi\|_{L^1(A)} + \|E^\varepsilon\|_{L^\infty(\mathbb{R}^+;L^{1+2/d}(\mathbb{T}^d))}\|\nabla_v^2\varphi\|_{L^1(A)}\right\} \\
\leq{}& (\bar\tau^2 + 3\bar\tau\eta + 3\eta^2)|A|(2\pi)^{-d}\|E^\varepsilon\|_{L^\infty(\mathbb{R}^+;L^{1+d/2}(\mathbb{T}^d))} \\
& \left\{\|\nabla_x E^\varepsilon\|_{L^\infty(\mathbb{R}^+;L^{1+2/d}(\mathbb{T}^d))}\|\nabla_v\varphi\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^d)} + \|E^\varepsilon\|_{L^\infty(\mathbb{R}^+;L^{1+2/d}(\mathbb{T}^d))}\|\nabla_v^2\varphi\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^d)}\right\} \\
\lesssim{}& \delta,
\end{aligned}$$

which shows uniform equi-integrability of the family g^ε. It remains to prove (74). Using Fourier series in the space variables, the matrix (71) rewrites as

$$\mathscr{D}^\varepsilon(t,v) = \frac{1}{\bar\tau}\int_0^{\bar\tau} ds \int_{-\eta}^{s} d\sigma \sum_{k\in\mathbb{Z}^d} e^{\mathrm{i}k\cdot v(\sigma+\eta)}\hat{E}^\varepsilon(t+s\varepsilon^2, k)\otimes\hat{E}^\varepsilon(t+(s-\eta-\sigma)\varepsilon^2, -k). \tag{75}$$

From (50), there exists $\lambda > 0$ such that

$$|\hat{E}^\varepsilon(t,k)| \leq c_0(1+|k|)^{-(1+\lambda)}. \tag{76}$$

Using (75) and (76) and weak compactness, we obtain (74), which ends the Proof of Lemma 3.

If we now assume

$$\lim_{\varepsilon \to 0} \mathcal{M}^{\varepsilon}(\varphi) = 0, \tag{77}$$

then, using Lemma 3 and (67), we obtain from (65) and (70) the diffusion equation (57) with

$$\begin{aligned} \mathscr{D}(t,v) &:= \lim_{\varepsilon\to 0} \frac{1}{\bar{\tau}} \int_0^{\bar{\tau}} ds \int_{-\eta}^{s} d\sigma \fint dx\, E^{\varepsilon}(t+s\varepsilon^2, x) \otimes E^{\varepsilon}(t+(s-\eta-\sigma)\varepsilon^2, x - v(\sigma+\eta)) \\ &= \lim_{\varepsilon\to 0} \frac{1}{\bar{\tau}} \int_0^{\bar{\tau}} ds \int_{-\eta}^{s} d\sigma \sum_{k\in\mathbb{Z}^d} e^{\mathrm{i}k\cdot v(\sigma+\eta)} \hat{E}^{\varepsilon}(t+s\varepsilon^2, k) \otimes \hat{E}^{\varepsilon}(t+(s-\eta-\sigma)\varepsilon^2, k)^*. \end{aligned} \tag{78}$$

Few remarks are now in order.

Remark 10 (open issues).

1. *In order to justify (77), we have to prove that the term*

$$\nabla_v \cdot \left(E^{\varepsilon}(t+(s-\eta-\sigma)\varepsilon^2, x) \otimes E^{\varepsilon}(t+s\varepsilon^2, x+v(\sigma+\eta)) \nabla_v \varphi(t,v) \right)$$

converges strongly in $L^1(\mathbb{R}_t^+ \times \mathbb{R}_s^+ \times \mathbb{R}_\sigma^+ \times Q)$ *as* $\varepsilon \to 0$.

2. *Show that* $\mathcal{T}_1^{\varepsilon} \rightharpoonup 0$ *in* $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d)$ *remains an open issue. Nevertheless, we may expect that there exist some mixing-type hypotheses, which could justify such limit.*
3. *Taking* $\hat{\theta} = \varepsilon^2 \bar{\tau}$ *in Eq. (66), the term* $\mathcal{T}_1^{\varepsilon}$ *is reminiscent of the term (109) appearing in the case of the non-self-consistent stochastic electric field (see Sec. IV). Let us note that in the stochastic case, without additional regularity assumptions on weak solutions, this term vanishes by using a time decorrelation property between* E^{ε} *and* f^{ε} *and by using the fact that the stochastic average of* E^{ε} *vanishes.*
4. *The parameter* $\bar{\tau}$ *is reminiscent of the autocorrelation time of particles* $\bar{\tau}$*, which is introduced in case of the non-self-consistent stochastic electric field (see Sec. IV).*

An explicit form of the diffusion matrix in the non-self-consistent deterministic case.

Without being able to prove some time decorrelation properties for the electric field E^{ε}, it is difficult to deduce the structure of the diffusion matrix $\mathscr{D}$. However, as in Sec. III B 1, we can design a non-self-consistent deterministic electric field by using the WKB expansion (58) to obtain an explicit diffusion matrix. This gives a formal example for which the diffusion matrix (78) is not zero and non-negative. Using the WKB approximation (59), the notation

$$\Delta\Omega := \partial_t \Omega(t,k) - k \cdot v,$$

and the parity of functions $\Omega(t,k)$ and $\hat{E}_0(t,k)$ in the variable k, we obtain from similar computations leading to (60),

$$\begin{aligned} \mathscr{D}(t,v) &= \lim_{\varepsilon\to 0} \frac{1}{\bar{\tau}} \int_0^{\bar{\tau}} ds \int_{-\eta}^{s} d\sigma \sum_{k\in\mathbb{Z}^d} e^{\mathrm{i}k\cdot v(\sigma+\eta)} \hat{E}^{\varepsilon}(t+s\varepsilon^2, k) \otimes \hat{E}^{\varepsilon}(t+(s-\eta-\sigma)\varepsilon^2, k)^* \\ &= \sum_{k\in\mathbb{Z}^d} \frac{1}{\bar{\tau}} \int_0^{\bar{\tau}} ds \int_0^{s+\eta} d\sigma e^{-\mathrm{i}\sigma\Delta\Omega(t,k)} \hat{E}_0(t,k) \otimes \hat{E}_0(t,k) \\ &= \sum_{k\in\mathbb{Z}^d} \left(\frac{1}{\mathrm{i}\Delta\Omega} - \frac{e^{-\mathrm{i}\eta\Delta\Omega}}{\mathrm{i}\Delta\Omega} \frac{1-e^{-\mathrm{i}\bar{\tau}\Delta\Omega}}{\mathrm{i}\bar{\tau}\Delta\Omega} \right) \hat{E}_0(t,k) \otimes \hat{E}_0(t,k) \\ &= \sum_{k\in\mathbb{Z}^d} \frac{\sin(\bar{\tau}\Delta\Omega/2)}{\bar{\tau}\Delta\Omega/2} \frac{\sin((\bar{\tau}/2+\eta)\Delta\Omega)}{\Delta\Omega} \hat{E}_0(t,k) \otimes \hat{E}_0(t,k). \end{aligned} \tag{79}$$

At this point, we obtain some important limits with respect to the parameters η and $\bar{\tau}$. The parameter $\bar{\tau}$ is the same as the one that we have defined in Sec. II and used in Sec. IV. Hence, parameters $\bar{\tau}$ and η can be seen as normalized particle autocorrelation times.

The first significant limit is $\eta \to +\infty$. Indeed, using the limit $\lim_{\eta\to+\infty} \sin(\eta\tau)/\tau = \pi\delta(\tau)$ in $\mathcal{D}'(\mathbb{R})$ and taking the limit $\eta \to +\infty$ in (79), we recover the same diffusion matrix (61) that we have obtained with the global-in-time approach of Sec. III B 1. Therefore, we recover the quasilinear diffusion matrix (62) too. In a sense, the limit $\eta \to +\infty$ corresponds to take into account all the past of the distribution function and particularly the initial condition as it was done in the global-in-time approach of Sec. III B 1. Therefore, it is consistent to obtain the result of the global-in-time approach by taking the limit $\eta \to +\infty$ in the local-in-time approach.

The second significant limit is $\eta \to 0$. Indeed, for $\eta = 0$, we obtain from (79) the following non-negative diffusion matrix:

$$\mathscr{D}(t,v) = \frac{\bar{\tau}}{2}\sum_{k\in\mathbb{Z}^d}\left(\frac{\sin\left(\frac{\bar{\tau}}{2}(\partial_t\Omega(t,k) - k\cdot v)\right)}{\frac{\bar{\tau}}{2}(\partial_t\Omega(t,k) - k\cdot v)}\right)^2 \hat{E}_0(t,k)\otimes\hat{E}_0(t,k). \tag{80}$$

This diffusion matrix seems more regular in velocity than the quasilinear diffusion matrix (62). This regularity improvement in velocity is reminiscent of the finite-$\bar{\tau}$ effect that we observe in the framework of the non-self-consistent stochastic electric field (see Sec. IV) and also in the resonance broadening like theory (see Sec. IV D 1).

The last significant limit is to keep η finite and to pass to the limit $\bar{\tau}\to+\infty$ in (79) or (80). In this limit, we also recover the quasilinear diffusion matrix (61) or (62), and Remark 9 still holds true. This result is consistent with the developements of Sec. IV D 2 for the non-self-consistent stochastic electric field.

Remark 11. As in Remark 9, if the electric field E^ε derives from a potential Φ^ε, then in the diffusion matrices (79) and (80), we should replace the matrix $\hat{E}_0(t,k)\otimes\hat{E}_0(t,k)$ by the matrix $|\hat{E}_0(t,k)|^2\, k\otimes k/|k|^2$.

IV. THE NON-SELF-CONSISTENT STOCHASTIC CASE

In this section, we deal with the non-self-consistent stochastic case. Before stating our main Theorem 2 in Sec. IV B, we start by describing the features of the stochastic electric field in Sec. IV A.

A. The turbulent electric field

Here, electrostatic turbulence is modeled through the random vector field E^ε. Let $(\mathscr{O},\mathcal{F},\mathbb{P})$ be a probability space, with $\mathbb{P}$ being a σ-finite measure. A random vector F is real vector-valued function defined on $\mathscr{O}$. When $F:\mathscr{O}\to\mathbb{R}^d$ is an integrable random vector, its expectation is given by

$$\mathbb{E}[F] = \int_{\mathscr{O}} \mathrm{d}\mathbb{P}(\varpi)\, F(\varpi).$$

From considerations of Sec. II A, the turbulent electric field E^ε has two time scales, one slow and the other fast. We then choose a turbulent electric field E^ε given by

$$E^\varepsilon(t,x) = E(t,t/\varepsilon^2,x;\varpi), \tag{81}$$

where the integrable random vector field E satisfies the following "stochastic" assumptions:

(H1) The random vector field E is centered, i.e.,

$$\mathbb{E}[E(t,\tau,x)] = 0,\quad \forall(t,\tau,x)\in\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{T}^d.$$

(H2) There exists a constant $\bar{\tau}>0$ such that for every $x,y\in\mathbb{R}^d$ and for every $\tau,\sigma\in\mathbb{R}^+$, the electric fields $E(t,\tau,x)$ and $E(s,\sigma,y)$ are independent random vector fields as soon as $|\tau-\sigma|\geq\bar{\tau}$. The autocorrelation time $\bar{\tau}$ is supposed fixed and finite, hence independent of ε.

(H3) There exists a matrix-valued function $\mathcal{R}_{\bar{\tau}}:\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}\times\mathbb{T}^d\to\mathbb{R}^{2d}$, called the autocorrelation matrix or the Reynolds electric stress tensor, such that

$$\mathbb{E}[E(t,\tau,x)\otimes E(s,\sigma,y)] = \mathcal{R}_{\bar{\tau}}(t,s,\tau-\sigma,x-y). \tag{82}$$

Hypothesis (H1) sets the stochastic average of E^ε to zero, which is standard and not restrictive. Assumption (H2) means that the turbulent electric field E^ε is time decorrelated on a time scale $\varepsilon^2\bar{\tau}$. Assumption (H2) can be seen as a hypothesis of propagation of "stochasticity" or propagation of independence of random vector fields. Therefore, two evaluations in time of the electric field, separated by a lapse of time larger than $\varepsilon^2\bar{\tau}$, are independent random vector fields. Hypothesis (H3) is the standard spatiotemporal homogeneity property of the turbulence, i.e., the spatiotemporal autocorrelation of the electric field E^ε is invariant under space and time translations. These assumptions are similar to the ones of Ref. 50.

Remark 12. In the nonlinear regime, the property of time decorrelation seems to be the cornerstone of the diffusion process for both the self-consistent and the non-self-consistent setting.[3,5,21] *An open and very difficult problem remains to show mathematically such time decorrelation property from only the deterministic Vlasov–Poisson system [(7) and (8)] and random initial data f_0. In a sense, this is what has been shown numerically in Refs. 10 and 11. This property of propagation of "stochasticity" is reminiscent of the property of propagation of chaos in statistical mechanics.*

In order to justify rigorously the diffusion limit, the stochastic electric field requires the following regularity assumptions:

(H4) The regularity of E is such that

$$E \in L^\infty\Big(\mathbb{R}^+ \times \mathbb{R}^+; W^{2,\infty}(\mathbb{T}^d)\Big) \quad \text{and} \quad \mathbb{E}\big[\|E\|^3_{L^\infty(\mathbb{R}^+\times\mathbb{R}^+;W^{2,\infty}(\mathbb{T}^d))}\big] =: C_E < \infty.$$

Assumption (H4) imposes the regularity (especially in space) of the random vector field E. It is worthwhile to end this section by giving an explicit example of a random field E, which satisfies assumptions (H1)–(H4). Following the spirit of Example 2 in Ref. 50, we construct in the Appendix a random vector field E satisfying these requirements.

B. Main theorem

Concerning the non-self-consistent stochastic case, we establish the following theorem:

Theorem 2. *Let E be an integrable random vector field satisfying assumptions (H1)–(H4), and let E^ε be given by (81). Let $\{f_0^\varepsilon\}_{\varepsilon>0}$ be a sequence of independent random non-negative initial data and C_0 be a positive constant such that for a.e. $\varpi \in \mathcal{O}$, $\|f_0^\varepsilon\|_{L^1(Q)} + \|f_0^\varepsilon\|_{L^\infty(Q)} \le C_0 < \infty$. Let $\mathcal{D}_{\bar\tau} = \mathcal{D}_{\bar\tau}(t,v)$ be the matrix-valued function defined by*

$$\mathcal{D}_{\bar\tau}(t,v) = \int_0^{\bar\tau} d\sigma\, \mathcal{R}_{\bar\tau}(t,t,\sigma,\sigma v), \tag{83}$$

the properties of which are described in Proposition 1. Let f^ε be the unique weak solution of the Vlasov equation (17), with initial data $f^\varepsilon_{|_{t=0}} = f_0^\varepsilon$.

Then, up to extraction of a subsequence, $\mathbb{E}[f_0^\varepsilon]$ converges in $L^\infty(Q)$ weak−∗ to a function $f_0 \in L^1 \cap L^\infty(Q)$, $\mathbb{E}[f^\varepsilon]$ converges in $L^\infty(\mathbb{R}^+; L^\infty(Q))$ weak−∗ to a function $f \in L^\infty(\mathbb{R}^+; L^1 \cap L^\infty(\mathbb{R}^d))$, and $\mathbb{E}[⨍dx\, f^\varepsilon]$ converges in $L^\infty(\mathbb{R}^+; L^\infty(\mathbb{R}^d))$ weak−∗ to f. Moreover $\mathbb{E}[⨍dx\, f^\varepsilon]$ converges in $\mathcal{C}(0,T; L^p(\mathbb{R}^d) - weak)$ to f for $1 < p < \infty$ and for all $T > 0$. The limit point $f = f(t,v)$ is solution of the following diffusion equation in the sense of distributions:

$$\partial_t f - \nabla_v \cdot (\mathcal{D}_{\bar\tau} \nabla_v f) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d), \qquad f_{|_{t=0}} = ⨍ dx\, f_0. \tag{84}$$

The Proof of Theorem 2 is postponed to Sec. IV C. General properties of the diffusion matrix $\mathcal{D}_{\bar\tau}$ and the autocorrelation matrix $\mathcal{R}_{\bar\tau}$ of Theorem 2 are stated in the following proposition:

Proposition 1 (properties of the diffusion matrix $\mathcal{D}_{\bar\tau}$). Under assumptions (H1)–(H4), the matrix-valued function $\mathcal{R}_{\bar\tau} : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{T}^d \to \mathbb{R}^{2d}$ and the diffusion matrix $\mathcal{D}_{\bar\tau} : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^{2d}$ satisfy the following properties:

- (i) $\mathcal{R}_{\bar\tau}(t,t,\tau,x) = \mathcal{R}^T_{\bar\tau}(t,t,-\tau,-x)$ *and* $\mathcal{R}_{\bar\tau}(t,t,\tau,x+2\pi k) = \mathcal{R}_{\bar\tau}(t,t,\tau,x), \quad \forall k \in \mathbb{Z}$.
- (ii) $\mathcal{R}_{\bar\tau} \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}; W^{2,\infty}(\mathbb{T}^d))$ *and* $\operatorname{supp}(\mathcal{R}_{\bar\tau}) \subset \mathbb{R}^+ \times \mathbb{R}^+ \times [-\bar\tau, \bar\tau] \times \mathbb{T}^d$.
- (iii) $\mathcal{D}_{\bar\tau} \in L^\infty(\mathbb{R}^+; W^{2,\infty}(\mathbb{R}^d))$ *and* $\operatorname{supp}(\mathcal{D}_{\bar\tau}) \subset \mathbb{R}^+ \times \mathbb{R}^d$.
- (iv) *The symmetric part of $\mathcal{D}_{\bar\tau}$ is non-negative, i.e., $X^T \mathcal{D}_{\bar\tau} X \ge 0$, $\forall \in X \in \mathbb{R}^d$.*

Proof. We start with property (i). Using (H3), we obtain

$$\mathcal{R}_{\bar\tau\, ij}(t,t,\tau-\sigma,x-y) = \mathbb{E}[E_i(t,\tau,x)E_j(t,\sigma,y)] = \mathbb{E}[E_j(t,\sigma,y)E_i(t,\tau,x)] = \mathcal{R}_{\bar\tau\, ji}(t,t,-(\tau-\sigma),-(x-y)),$$

and for all $k, k' \in \mathbb{Z}$,

$$\begin{aligned}\mathcal{R}_{\bar\tau}(t,t,\tau-\sigma,x-y) &= \mathbb{E}[E(t,\tau,x)\otimes E(t,\sigma,y)] = \mathbb{E}[E(t,\tau,x+2\pi k)\otimes E(t,\sigma,y+2\pi k')] \\ &= \mathcal{R}_{\bar\tau\, ij}(t,t,\tau-\sigma,x-y+2\pi(k-k')).\end{aligned}$$

The regularity property (ii) for $\mathcal{R}_{\bar\tau}$ comes immediately from the regularity assumption (H4) for E and the definition (H3) for $\mathcal{R}_{\bar\tau}$. The regularity property (iii) for $\mathcal{D}_{\bar\tau}$ is the straight consequence of the definition (83) for $\mathcal{D}_{\bar\tau}$ and the regularity property (ii) for $\mathcal{R}_{\bar\tau}$. The support of $\mathcal{D}_{\bar\tau}$ is obvious, while the support of $\mathcal{R}_{\bar\tau}$ results from assumptions (H1)–(H3). Indeed, if $|\tau-\sigma| > \bar\tau$, then (H1)–(H3) imply that $\mathcal{R}_{\bar\tau}(t,t,\tau-\sigma,x-y) = \mathbb{E}[E(t,\tau,x)\otimes E(t,\sigma,y)] = \mathbb{E}[E(t,\tau,x)]\otimes\mathbb{E}[E(t,\sigma,y)] = 0$. We end with the property (iv). Using properties (i) and (ii), we obtain

$$X^T \mathscr{D}_{\bar{\tau}} X = \sum_{i,j} X_i X_j \int_0^{\bar{\tau}} d\sigma\, \mathcal{R}_{\bar{\tau}\, ij}(t,t,\sigma,\sigma v) = \frac{1}{2}\sum_{i,j} X_i X_j \int_{-\bar{\tau}}^{\bar{\tau}} d\sigma\, \mathcal{R}_{\bar{\tau}\, ij}(t,t,\sigma,\sigma v)$$
$$= \frac{1}{2}\sum_{i,j} X_i X_j \int_{-\infty}^{+\infty} d\sigma\, \mathcal{R}_{\bar{\tau}\, ij}(t,t,\sigma,\sigma v). \tag{85}$$

Using Lemma 3.1 of Ref. 50, which states that for all $g \in L^1(\mathbb{R})$, we have

$$\int_{-\infty}^{+\infty} g(s) ds = \lim_{R\to+\infty} \frac{1}{2R} \int_{-R}^{R} ds \int_{-R}^{R} dt\, g(s-t),$$

we obtain from (85),

$$X^T \mathscr{D}_{\bar{\tau}} X = \lim_{R\to+\infty} \frac{1}{4R} \sum_{i,j} X_i X_j \int_{-R}^{R} d\sigma \int_{-R}^{R} d\theta\, \mathbb{E}[E_i(t,\sigma,\sigma v) E_j(t,\theta,\theta v)]$$
$$= \lim_{R\to+\infty} \frac{1}{4R} \int_{-R}^{R} d\sigma \int_{-R}^{R} d\theta\, \mathbb{E}[X\cdot E(t,\sigma,\sigma v)\, X\cdot E(t,\theta,\theta v)]$$
$$= \lim_{R\to+\infty} \frac{1}{4R} \mathbb{E}\left[\left(\int_{-R}^{R} d\sigma\, X\cdot E(t,\sigma,\sigma v)\right)^2\right] \geq 0,$$

which ends the Proof of Proposition 1. □

To end this section, it is worthwhile to compare the diffusion matrix obtained here for the stochastic case and the one obtained for the deterministic case in Sec. III B. If ε is small enough such that $t/\varepsilon^2 > \bar{\tau}$, then using (H3), (81), and the compact support of $\mathcal{R}_{\bar{\tau}}(\cdot,\cdot,\sigma,\cdot)$ in the variable σ (included in $[-\bar{\tau},\bar{\tau}]$; see Proposition 1), the stochastic diffusion matrix $\mathscr{D}_{\bar{\tau}}$, defined by (83), rewrites as

$$\mathscr{D}_{\bar{\tau}}(t,v) = \lim_{\varepsilon\to 0} \int_0^{\bar{\tau}} d\sigma\, \mathbb{E}[E^\varepsilon(t,x)\otimes E^\varepsilon(t-\varepsilon^2\sigma, x-\sigma v)]$$
$$= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^+} d\sigma\, \chi_{[0,t/\varepsilon^2]}(\sigma) \mathbb{E}[E^\varepsilon(t,x)\otimes E^\varepsilon(t-\varepsilon^2\sigma, x-\sigma v)]. \tag{86}$$

Comparing the deterministic diffusion matrix (56) (obtained for the global-in-time approach in Sec. III B 1) and stochastic diffusion matrix (86), we observe that they are the same except that the space average is replaced by the statistical average. If we suppose that in definition (56) of the deterministic diffusion matrix, the electric field E^ε is a random vector field and if we take the expectation value of (56), then the homogeneity property (H3) implies that the space average is trivially the identity. Therefore, we recover the stochastic diffusion matrix (86) or (83) from the "deterministic" one (56) by a statistical average and the homogeneity property. The link between the deterministic and stochastic diffusion matrices is also reinforced by the following corollary:

Corollary 1. Let $(t,k)\mapsto \Omega(t,k) := \omega(k)t$ be a real-valued function, where the given real-valued function $k\mapsto\omega(k)$ is odd in the variable k. Let $(t,k)\mapsto \hat{E}_0(t,k)$ be a given real vector-valued function, which is even in the variable k and such that $|k||\hat{E}_0| \in L^\infty(\mathbb{R}^+;\ell^2(\mathbb{Z}^d))$. Then, there exists a matrix-valued function $\mathcal{R}^0_{\bar{\tau}} : \mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}\times\mathbb{T}^d \to \mathbb{R}^{2d}$ such that the associated diffusion matrix, defined by formula (83) of Theorem 2, is

$$\mathscr{D}_{\bar{\tau}}(t,v) = \frac{\bar{\tau}}{2}\sum_{k\in\mathbb{Z}^d}\left(\frac{\sin\left(\frac{\bar{\tau}}{2}(\partial_t\Omega(t,k)-k\cdot v)\right)}{\frac{\bar{\tau}}{2}(\partial_t\Omega(t,k)-k\cdot v)}\right)^2 \hat{E}_0(t,k)\otimes\hat{E}_0(t,k). \tag{87}$$

Moreover, the autocorrelation matrix $\mathcal{R}^0_{\bar{\tau}}$ and its associated diffusion matrix (87) satisfy properties (i)–(iv) of Proposition 1.

Before giving the proof of Corollary 1, we observe that the diffusion matrix (87) is the same as the diffusion matrix (80), obtained for the local-in-time approach of the non-self-consistent deterministic case in Sec. III B 2.

Proof of Corollary 1. We consider the Fourier series decomposition of a given smooth electric field E,

$$E(t,\tau,x) = \sum_{k\in\mathbb{Z}^d} e^{\mathrm{i}k\cdot x}\hat{E}(t,\tau,k),$$

where, without loss of generality, we take

$$\hat{E}(t,\tau,k) = \hat{E}_0(t,\tau,k)e^{-\mathrm{i}\Omega(\tau,k)} = \hat{E}_0(t,\tau,k)e^{-\mathrm{i}\omega(k)\tau},$$

with the function $\Omega(t,k)$ [respectively, $\omega(k)$] chosen like in the statement of Corollary 1. Since $\hat{E}(t,\tau,k)$ is Hermitian [i.e., $\hat{E}^*(t,\tau,k) = \hat{E}(t,\tau,-k)$] and because the function $k \mapsto \omega(k)$ is odd, we could choose either $\hat{E}_0(t,\tau,k)$ Hermitian or real and even in k. We restrict ourselves to the case where $\hat{E}_0(t,\tau,k)$ is real and even in k. In order to be consistent with hypothesis (H3), using the real vector-valued function $\hat{E}_0(t,k)$ like in the statement of Corollary 1, we can choose $\hat{E}_0(t,\tau,k)$ such that

$$\mathbb{E}\big[\hat{E}_0(t,\tau,k) \otimes \hat{E}_0(t,\sigma,k')\big] = \hat{E}_0(t,k) \otimes \hat{E}_0(t,k) A_{\bar{\tau}}(\tau-\sigma)\delta(k+k'), \tag{88}$$

where $s \mapsto A_{\bar{\tau}}(s) : \mathbb{R} \to \mathbb{R}^+$ is a real non-negative even bounded function. From (88), we obtain

$$\mathbb{E}[E(t,\tau,x) \otimes E(t,\sigma,y)] = \sum_{k\in\mathbb{Z}^d} \hat{E}_0(t,k) \otimes \hat{E}_0(t,k)\, A_{\bar{\tau}}(\tau-\sigma)e^{\mathrm{i}k\cdot(x-y)}e^{-\mathrm{i}\omega(k)(\tau-\sigma)},$$

and then from (82), we obtain

$$\mathcal{R}^0_{\bar{\tau}}(t,t,\tau,x) = \sum_{k\in\mathbb{Z}^d} \hat{E}_0(t,k) \otimes \hat{E}_0(t,k)\, A_{\bar{\tau}}(\tau)e^{-\mathrm{i}(\omega(k)\tau - k\cdot x)}. \tag{89}$$

The autocorrelation matrix $\mathcal{R}^0_{\bar{\tau}}$ given by (89) satisfies the property (*i*) of Proposition 1. Setting the resonance function

$$R_{\bar{\tau}}(\xi) := \frac{\bar{\tau}}{2}\left(\frac{\sin(\bar{\tau}\xi/2)}{\bar{\tau}\xi/2}\right)^2,$$

we define the time autocorrelation function $A_{\bar{\tau}}$ as the inverse Fourier transform of the function $2R_{\bar{\tau}}$, i.e.,

$$A_{\bar{\tau}}(\sigma) := \frac{1}{\pi}\int_{-\infty}^{+\infty} e^{\mathrm{i}\xi\sigma}R_{\bar{\tau}}(\xi)d\xi = \Lambda(\sigma/\bar{\tau}), \tag{90}$$

where the function $s \mapsto \Lambda(s)$ is the triangular function of support $[-1,1]$. As a consequence, $\|A_{\bar{\tau}}\|_{L^\infty(\mathbb{R})} \le 1$, and using $|k\|\hat{E}_0| \in L^\infty(\mathbb{R}^+;\ell^2(\mathbb{Z}^d))$, we obtain that $\mathcal{R}^0_{\bar{\tau}} \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}; W^{2,\infty}(\mathbb{T}^d))$. In addition, from (90), we deduce that $\operatorname{supp}(\mathcal{R}^0_{\bar{\tau}}) \subset \mathbb{R}^+ \times \mathbb{R}^+ \times [-\bar{\tau},\bar{\tau}] \times \mathbb{T}^d$. Therefore, the autocorrelation matrix $\mathcal{R}^0_{\bar{\tau}}$ given by (89) and (90) satisfies the property (*ii*) of Proposition 1. Finally, let us compute the diffusion matrix $\mathcal{D}_{\bar{\tau}}$ from (83), (89), and (90). Using parity properties of the functions ω, $A_{\bar{\tau}}$, and $\hat{E}_0$, we obtain

$$\begin{aligned}
\mathcal{D}_{\bar{\tau}}(t,v) &= \int_0^{\bar{\tau}} d\sigma\, \mathcal{R}^0_{\bar{\tau}}(t,t,\sigma,\sigma v)\\
&= \sum_{k\in\mathbb{Z}^d} \hat{E}_0(t,k) \otimes \hat{E}_0(t,k) \int_0^{\bar{\tau}} d\sigma\, e^{\mathrm{i}(\omega(k)-k\cdot v)\sigma} A_{\bar{\tau}}(\sigma)\\
&= \sum_{k\in\mathbb{Z}^d} \hat{E}_0(t,k) \otimes \hat{E}_0(t,k) \int_{\mathbb{R}^+} d\sigma\, e^{\mathrm{i}(\omega(k)-k\cdot v)\sigma} \Lambda(\sigma/\bar{\tau})\\
&= \frac{1}{2}\sum_{k\in\mathbb{Z}^d} \hat{E}_0(t,k) \otimes \hat{E}_0(t,k) \int_{\mathbb{R}} d\sigma\, e^{\mathrm{i}(\omega(k)-k\cdot v)\sigma} \Lambda(\sigma/\bar{\tau})\\
&= \frac{\bar{\tau}}{2}\sum_{k\in\mathbb{Z}^d} \hat{E}_0(t,k) \otimes \hat{E}_0(t,k) \left(\frac{\sin\left(\frac{\bar{\tau}}{2}(\omega(k)-k\cdot v)\right)}{\frac{\bar{\tau}}{2}(\omega(k)-k\cdot v)}\right)^2,
\end{aligned} \tag{91}$$

which is (87). By a direct differentiation of (91) with respect to v, we verify easily the property (*iii*) of Proposition 1, while the property (*iv*) of Proposition 1 is obvious from the structure of (91).

C. Proof of theorem 2

Let us rewrite the Vlasov equation (17) in the following form:

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$$\partial_t f^\varepsilon + \frac{1}{\varepsilon^2}\mathcal{L} f^\varepsilon = \mathcal{N}_t^\varepsilon f^\varepsilon, \tag{92}$$

$$f^\varepsilon_{|_{t=0}} = f_0^\varepsilon, \tag{93}$$

where the linear operators $\mathcal{L}$ and $\mathcal{N}_t^\varepsilon$ are defined by

$$\mathcal{L} = v\cdot\nabla_x, \qquad \mathcal{N}_t^\varepsilon = -\frac{1}{\varepsilon}E^\varepsilon(t,x)\cdot\nabla_v = -\frac{1}{\varepsilon}E(t,t/\varepsilon^2,x)\cdot\nabla_v. \tag{94}$$

Obviously, the operators $\mathcal{L}$ and $\mathcal{N}_t^\varepsilon$ are skew-adjoint for the scalar product of $L^2(Q)$, while the operator $\mathcal{L}$ and the deterministic group S_t^ε, generated by $\varepsilon^{-2}\mathcal{L}$ (see Sec. II B), commute with $\mathbb{E}$. Of course, space and statistical averages commute. From hypothesis (H2), the random operators $\mathcal{N}_t^\varepsilon$ and $\mathcal{N}_s^\varepsilon$ are independent as soon as $|t-s| > \varepsilon^2\bar{\tau}$.

The next useful proposition states that time decorrelation of the stochastic electric field also entails time decorrelation between the distribution function and the electric field.

Proposition 2 (time decorrelation property between f^ε and E^ε). Assume (H2). *Suppose that the random initial data f_0^ε and the electric field E^ε are independent. Then, $\mathcal{N}_s^\varepsilon$ is independent of $f^\varepsilon(t)$ as soon as $s \ge t + \varepsilon^2\bar{\tau}$.*

Proof. From the Duhamel formula

$$f^\varepsilon(t) = S_t^\varepsilon f_0^\varepsilon + \int_0^t d\sigma\, S_{t-\sigma}^\varepsilon \mathcal{N}_\sigma^\varepsilon f^\varepsilon(\sigma),$$

where $t\mapsto S_t^\varepsilon$ is the deterministic group generated by $\varepsilon^{-2}\mathcal{L}$ (see Sec. II B), we observe that $f^\varepsilon(t)$ depends only on f_0^ε and $\mathcal{N}_\sigma^\varepsilon$ [or $E^\varepsilon(\sigma,\cdot)$] for $\sigma \le t$. Since f_0^ε is independent of $E^\varepsilon(t,\cdot)$ $\forall t\in\mathbb{R}$, and since the electric fields $E^\varepsilon(s,\cdot)$ and $E^\varepsilon(t,\cdot)$ are independent as soon as $s > t+\varepsilon^2\bar{\tau}$ [assumption (H2)], we obtain from the Duhamel formula the desired result. □

We start our analysis by recalling basic statements that we collect in the following proposition:

Proposition 3. Assume (H4) *and consider a sequence $\{f_0^\varepsilon\}_{\varepsilon>0}$ of initial data such that*

$$f_0^\varepsilon \ge 0, \quad \text{and for a.e. } \varpi\in\mathcal{O},\ \|f_0^\varepsilon\|_{L^1(Q)} + \|f_0^\varepsilon\|_{L^\infty(Q)} \le C_0 < \infty.$$

Then, for any $\varepsilon>0$, the Cauchy's problem (92) and (93) has a unique non-negative solution $f^\varepsilon\in\mathscr{C}(\mathbb{R}^+, L^1\cap L^\infty(Q))$, which is given by

$$f^\varepsilon(t,x,v) = f_0^\varepsilon(X^\varepsilon(0;t,x,v), V^\varepsilon(0;t,x,v)), \tag{95}$$

where the characteristic curves $(X^\varepsilon, V^\varepsilon)$ are solutions to the ODEs,

$$\frac{dX^\varepsilon}{dt}(t) = \frac{1}{\varepsilon^2}V^\varepsilon(t), \quad \frac{dV^\varepsilon}{dt}(t) = \frac{1}{\varepsilon}E^\varepsilon(t,X(t)), \quad X^\varepsilon(0;0,x,v) = x, \quad V^\varepsilon(0;0,x,v) = v. \tag{96}$$

Moreover, we have the a priori estimates

$$\|f^\varepsilon(t)\|_{L^p(Q)} = \|f_0^\varepsilon\|_{L^p(Q)}, \quad 1\le p\le\infty.$$

In addition, there exist a function $f_0\in L^1\cap L^\infty(\mathbb{R}^d)$ and a function $f\in L^\infty(\mathbb{R}^+;L^1\cap L^\infty(\mathbb{R}^d))$, such as, up to subsequences,

$$\mathbb{E}[f_0^\varepsilon] \rightharpoonup f_0 \ \ \text{in } L^\infty(Q) \ \ \text{weak}-* \quad \text{and} \quad \mathbb{E}[f^\varepsilon] \rightharpoonup f \ \ \text{in } L^\infty(\mathbb{R}^+;L^\infty(Q)) \ \ \text{weak}-*.$$

The limit point f is such that $\fint dx\, f = f\in L^\infty(\mathbb{R}^+;L^1\cap L^\infty(\mathbb{R}^d))$. The function $\mathbb{E}[\fint dx\, f^\varepsilon]$ is the solution of

$$\partial_t\mathbb{E}\Big[\fint dx\, f^\varepsilon\Big] + \nabla_v\cdot\mathbb{E}\Big[\fint dx\,\frac{E^\varepsilon f^\varepsilon}{\varepsilon}\Big] = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^+\times\mathbb{R}^d), \tag{97}$$

$$\mathbb{E}\Big[\fint dx\, f^\varepsilon\Big]_{|_{t=0}} = \mathbb{E}\Big[\fint dx\, f_0^\varepsilon\Big]. \tag{98}$$

Proof. Since $\|\mathbb{E}[f_0^\varepsilon]\|_{L^1(Q)} + \|\mathbb{E}[f_0^\varepsilon]\|_{L^\infty(Q)} \le \mathbb{E}[\|f_0^\varepsilon\|_{L^1(Q)}] + \mathbb{E}[\|f_0^\varepsilon\|_{L^\infty(Q)}] \le C_0 < \infty$, by weak compactness arguments, there exists a function $f_0\in L^1\cap L^\infty(Q)$ such that $\mathbb{E}[f_0^\varepsilon]$ (up to a subsequence) converges in $L^\infty(Q)$ weak-$*$ to f_0. Using the regularity hypothesis (H4) for the electric field E, the Cauchy–Lipschitz–Picard theorem for ODEs gives existence and uniqueness of a regular Lagrangian flow $(X^\varepsilon, V^\varepsilon)$, which is a solution of (96). It follows from standard results on first-order transport equations (see, e.g., Ref. 12) that the Lagrangian solution to (92) and (93) is given by (95). From (95), we obtain $\|\mathbb{E}[f^\varepsilon]\|_{L^\infty(Q)} = \|\mathbb{E}[f_0^\varepsilon]\|_{L^\infty(Q)} \le C_0 < \infty$. Moreover, using skew-adjointness of $\mathcal{L}$ and $\mathcal{N}_t^\varepsilon$, we can prove, following standard lines, that $\partial_t\|f^\varepsilon(t)\|_{L^p(Q)} = 0$ for $1 \le p \le \infty$. This leads to $\mathbb{E}[\|f^\varepsilon\|_{L^\infty(\mathbb{R}^+;L^p(Q))}] = \mathbb{E}[\|f_0^\varepsilon\|_{L^p(Q)}] \le C_0 < \infty$ and $\|\mathbb{E}[f^\varepsilon]\|_{L^\infty(\mathbb{R}^+;L^p(Q))} \le \mathbb{E}[\|f^\varepsilon\|_{L^\infty(\mathbb{R}^+;L^p(Q))}] \le C_0 < \infty$. By weak compactness arguments, there exists a function $f \in L^\infty(\mathbb{R}^+;L^1 \cap L^\infty(Q))$ such that $\mathbb{E}[f^\varepsilon]$ (up to a subsequence) converges in $L^\infty(\mathbb{R}^+;L^\infty(Q))$ weak$-*$ to f. Now, we claim that $\varepsilon^2\mathcal{N}_t^\varepsilon f^\varepsilon \to 0$ in $\mathcal{D}'(\mathbb{R}^+ \times Q)$ as ε tends to zero. Indeed, we have for all $\varphi \in \mathcal{D}(\mathbb{R}^+ \times Q)$,

$$|\langle \varepsilon^2\mathcal{N}_t^\varepsilon f^\varepsilon, \varphi\rangle| = \varepsilon^2|\langle f^\varepsilon, \mathcal{N}_t^{\varepsilon\,*}\varphi\rangle| \le \varepsilon\|f_0^\varepsilon\|_{L^\infty(Q)}\|\varphi\|_{L^\infty(\mathbb{R}^+;W^{1,1}(Q))}\|E\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{T}^d)} \to 0$$

as $\varepsilon \to 0$. Then, multiplying the Vlasov equation (92) by ε^2, taking its expectation value, and letting ε go to zero, we find

$$\mathcal{L}f = 0 \quad in\ \mathcal{D}'(\mathbb{R}^+ \times Q), \tag{99}$$

where we have used the commutation property between $\mathbb{E}$ and $\mathcal{L}$. From Lemma 1 and (99), we infer that f is independent of x and $⨍dx\, f = f \in L^\infty(\mathbb{R}^+;L^1 \cap L^\infty(\mathbb{R}^d))$. Finally, the Vlasov equation (92) is averaged in space and then rewritten in a weak form. Taking the expectation value of the result, we obtain (97) and (98). □

The rest of the proof is devoted to pass to the limit in Eq. (97). For this, we can first start with a simple iteration of the Duhamel formula as it was done in Sec. III B 2 for the deterministic case. As explained in Sec. IV C 1, this method fails. To solve this problem, we use the method of Ref. 50, which consists to apply a double iteration of the Duhamel formula for f. This is described in Sec. IV C 2.

1. Simple iteration of the Duhamel formula

Following what we have done in Sec. III B 2, where a simple iteration of the Duhamel formula is used, we obtain, for all $\varphi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^d)$,

$$\int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv\, \varphi(t,v) \frac{\mathbb{E}[⨍\, dx\, f^\varepsilon(t+\theta)] - \mathbb{E}[⨍\, dx\, f^\varepsilon(t)]}{\theta} = \mathcal{T}_1^\varepsilon(\varphi) + \mathcal{T}_2^\varepsilon(\varphi), \tag{100}$$

where

$$\mathcal{T}_1^\varepsilon(\varphi) := \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \int_t^{t+\theta} ds ⨍ dx\, \frac{1}{\varepsilon\theta}\mathbb{E}\big[E^\varepsilon(s)\cdot\nabla_v\varphi(t,v)S^\varepsilon_{s-t+\hat\theta}f^\varepsilon(t-\hat\theta)\big] \tag{101}$$

and

$$\mathcal{T}_2^\varepsilon(\varphi) := \mathcal{J}^\varepsilon(\varphi) + \mathcal{M}^\varepsilon(\varphi),$$

with

$$\mathcal{J}^\varepsilon(\varphi) := \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \int_t^{t+\theta} ds \int_{t-\hat\theta}^s d\sigma \frac{1}{\varepsilon^2\theta} f(t,v)\nabla_v \cdot \Big(⨍ dx\, \mathbb{E}\big[S^\varepsilon_{s-\sigma}E^\varepsilon(\sigma,x)\otimes E^\varepsilon(s,x)\nabla_v\varphi(t,v)\big]\Big) \tag{102}$$

and

$$\mathcal{M}^\varepsilon(\varphi) := \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \int_t^{t+\theta} ds \int_{t-\hat\theta}^s d\sigma \frac{1}{\varepsilon^2\theta}⨍ dx\, \mathbb{E}\big[\big(S^\varepsilon_{s-\sigma}f^\varepsilon(\sigma) - f(t,v)\big)\nabla_v\cdot\big(S^\varepsilon_{s-\sigma}E^\varepsilon(\sigma,x)\otimes E^\varepsilon(s,x)\nabla_v\varphi(t,v)\big)\big].$$

By choosing $\hat\theta \ge \varepsilon^2\bar\tau$, we can use the time decorrelation hypothesis (H2), Proposition 2, and assumption (H1) to show that for Eq. (101), we have $\mathcal{T}_1^\varepsilon(\varphi) = 0$. The term $\mathcal{J}^\varepsilon(\varphi)$, defined by Eq. (102), can be treated as it was done in Sec. III B and it gives the diffusion term. It remains to deal with the error term $\mathcal{M}^\varepsilon(\varphi)$, which is of order zero with respect to ε, i.e., $\mathcal{M}^\varepsilon(\varphi) = \mathcal{O}(\varepsilon^0)$. For this, we observe that we have to evaluate the expectation of a cubic product between $f^\varepsilon(\sigma) - f(t)$, $E^\varepsilon(\sigma)$ and $E^\varepsilon(s)$, i.e., schematically $\mathbb{E}[(f^\varepsilon(\sigma) - f(t))E^\varepsilon(\sigma)E^\varepsilon(s)]$. Using hypotheses (H2) and (H3) and Proposition 2, we would like to replace an expression of the form $\mathbb{E}[(f^\varepsilon(\sigma) - f(t))E^\varepsilon(\sigma)E^\varepsilon(s)]$ by an expression of the form $\mathbb{E}[f^\varepsilon - f]\mathbb{E}[E^\varepsilon E^\varepsilon]$ because from Proposition 3, we know that $\mathbb{E}[f^\varepsilon - f] \rightharpoonup 0$ in $L^\infty(\mathbb{R}^+; L^\infty(Q))$ weak−∗. Unfortunately, it is not possible, since $f^\varepsilon(\sigma)$ and $E^\varepsilon(\sigma)$ are evaluated at the same time σ. To remedy this problem, we follow the procedure of Ref. 50, which consists of using a double iteration of the Duhamel formula for f. A double instead of a simple iteration of the Duhamel formula is used to go back in time far enough in order to use the time decorrelation property (H2). The price of this procedure is the introduction of a second error term, namely, μ_t^ε [defined by (108)]. For the error term μ_t^ε, we face the same problem as for the term $\mathcal{M}^\varepsilon$, i.e., we cannot use the time decorrelation hypotheses (H2) and (H3) and Proposition 2. Nevertheless, the error term μ_t^ε is of the order $\mathcal{O}(\varepsilon)$. Therefore, to show that $\lim_{\varepsilon\to 0}\mu_t^\varepsilon = 0$ in the distributional sense, we do not use time decorrelation hypotheses (which are useless), but we appeal to the regularity hypotheses on the electric field E^ε, namely, (H4).

2. Double iteration of the Duhamel formula

First, we recall that $t \mapsto S_t^\varepsilon$ is the (deterministic) group on $L^p(Q)$, $1 \le p \le \infty$, generated by $\varepsilon^{-2}\mathcal{L}$ [see Eq. (20)]. Using the group S_t^ε and the Duhamel formula, the formal solution to (92) is given by

$$f^\varepsilon(t) = S_{t-s}^\varepsilon f^\varepsilon(s) + \int_s^t d\tau\, S_{t-\tau}^\varepsilon \mathcal{N}_\tau^\varepsilon f^\varepsilon(\tau). \tag{103}$$

Taking $s = t - \varepsilon^2\bar{\tau}$ in (103) and making the change of variable $\tau = t - \sigma$, we obtain from (103),

$$f^\varepsilon(t) = S_{\varepsilon^2\bar{\tau}}^\varepsilon f^\varepsilon(t - \varepsilon^2\bar{\tau}) + \int_0^{\varepsilon^2\bar{\tau}} d\sigma\, S_\sigma^\varepsilon \mathcal{N}_{t-\sigma}^\varepsilon f^\varepsilon(t-\sigma). \tag{104}$$

In the integral term of (104), we observe that the electric field and the distribution function are evaluated at the same time $t - \sigma$. As a consequence, if we substitute (104) to f^ε in the right-hand side of (97) (like it was done in Sec. IV C 1), we obtain a quadratic term with respect to the electric field that we cannot decorrelate in time from the distribution function. For this reason and following Ref. 50, we iterate a second time the Duhamel formula. In the same way that we obtained (103), we obtain

$$f^\varepsilon(t-\sigma) = S_{2\varepsilon^2\bar{\tau}-\sigma}^\varepsilon f^\varepsilon(t - 2\varepsilon^2\bar{\tau}) + \int_0^{2\varepsilon^2\bar{\tau}-\sigma} ds\, S_s^\varepsilon \mathcal{N}_{t-\sigma-s}^\varepsilon f^\varepsilon(t-\sigma-s). \tag{105}$$

Substituting the right-hand side of (105) to $f^\varepsilon(t-\sigma)$ in the right-hand side of (104) and using the properties of the group S_t^ε, we obtain

$$\begin{aligned} f^\varepsilon(t) = S_{\varepsilon^2\bar{\tau}}^\varepsilon f^\varepsilon(t - \varepsilon^2\bar{\tau}) &+ \int_0^{\varepsilon^2\bar{\tau}} d\sigma\, S_\sigma^\varepsilon \mathcal{N}_{t-\sigma}^\varepsilon S_{-\sigma}^\varepsilon S_{2\varepsilon^2\bar{\tau}}^\varepsilon f^\varepsilon(t - 2\varepsilon^2\bar{\tau}) \\ &+ \int_0^{\varepsilon^2\bar{\tau}} d\sigma \int_0^{2\varepsilon^2\bar{\tau}-\sigma} ds\, S_\sigma^\varepsilon \mathcal{N}_{t-\sigma}^\varepsilon S_s^\varepsilon \mathcal{N}_{t-\sigma-s}^\varepsilon f^\varepsilon(t-\sigma-s). \end{aligned} \tag{106}$$

Applying the operator $\mathcal{N}_t^\varepsilon$ to (106) and then applying successively the average in space and the expectation value, we obtain

$$\begin{aligned} -\nabla_v \cdot \mathbb{E}\left[⨍ dx\, \frac{E^\varepsilon(t) f^\varepsilon(t)}{\varepsilon}\right] = ⨍ dx\, \mathbb{E}\left[\mathcal{N}_t^\varepsilon S_{\varepsilon^2\bar{\tau}}^\varepsilon f^\varepsilon(t - \varepsilon^2\bar{\tau})\right] \\ + \int_0^{\varepsilon^2\bar{\tau}} d\sigma ⨍ dx\, \mathbb{E}\left[\mathcal{N}_t^\varepsilon S_\sigma^\varepsilon \mathcal{N}_{t-\sigma}^\varepsilon S_{-\sigma}^\varepsilon S_{2\varepsilon^2\bar{\tau}}^\varepsilon f^\varepsilon(t - 2\varepsilon^2\bar{\tau})\right] + \mu_t^\varepsilon, \end{aligned} \tag{107}$$

with

$$\mu_t^\varepsilon = \int_0^{\varepsilon^2\bar{\tau}} d\sigma \int_0^{2\varepsilon^2\bar{\tau}-\sigma} ds ⨍ dx\, \mathbb{E}\left[\mathcal{N}_t^\varepsilon S_\sigma^\varepsilon \mathcal{N}_{t-\sigma}^\varepsilon S_s^\varepsilon \mathcal{N}_{t-\sigma-s}^\varepsilon f^\varepsilon(t-\sigma-s)\right]. \tag{108}$$

Using Proposition 2, we obtain that $f^\varepsilon(t)$ is independent of $\mathcal{N}_s^\varepsilon$ as soon as $s \ge t + \varepsilon^2\bar{\tau}$. Then, using hypothesis (H1), we obtain

$$\mathbb{E}\left[\mathcal{N}_t^\varepsilon S_{\varepsilon^2\bar{\tau}}^\varepsilon f^\varepsilon(t - \varepsilon^2\bar{\tau})\right] = \mathbb{E}\left[\mathcal{N}_t^\varepsilon\right] S_{\varepsilon^2\bar{\tau}}^\varepsilon \mathbb{E}\left[f^\varepsilon(t - \varepsilon^2\bar{\tau})\right] = 0. \tag{109}$$

We note that the analysis of the term (109) is the same as the one done for the term (101). Therefore, the first term of the right-hand side of (107) vanishes. Since Proposition 2 implies that $\mathcal{N}_t^\varepsilon$ and $\mathcal{N}_{t-\sigma}^\varepsilon$ are independent of $f^\varepsilon(t-2\varepsilon^2\bar{\tau})$ for $0 \le \sigma \le \varepsilon^2\bar{\tau}$, we obtain from (107),

$$\begin{aligned} -\nabla_v \cdot \mathbb{E}\Big[\fint dx\, \frac{E^\varepsilon(t) f^\varepsilon(t)}{\varepsilon}\Big] &= \int_0^{\varepsilon^2\bar{\tau}} d\sigma \fint dx\, \mathbb{E}\big[\mathcal{N}_t^\varepsilon S_\sigma^\varepsilon \mathcal{N}_{t-\sigma}^\varepsilon S_{-\sigma}^\varepsilon\big]\mathbb{E}\big[S_{2\varepsilon^2\bar{\tau}}^\varepsilon f^\varepsilon(t-2\varepsilon^2\bar{\tau})\big] + \mu_t^\varepsilon \\ &= \int_0^{\varepsilon^2\bar{\tau}} d\sigma \fint dx\, \mathbb{E}\big[\mathcal{N}_t^\varepsilon S_\sigma^\varepsilon \mathcal{N}_{t-\sigma}^\varepsilon S_{-\sigma}^\varepsilon\big]\mathbb{E}\big[f^\varepsilon(t)\big] \\ &+ \int_0^{\varepsilon^2\bar{\tau}} d\sigma \fint dx\, \mathbb{E}\big[\mathcal{N}_t^\varepsilon S_\sigma^\varepsilon \mathcal{N}_{t-\sigma}^\varepsilon S_{-\sigma}^\varepsilon\big]\mathbb{E}\big[S_{2\varepsilon^2\bar{\tau}}^\varepsilon f^\varepsilon(t-2\varepsilon^2\bar{\tau}) - f^\varepsilon(t)\big] \\ &+ \mu_t^\varepsilon. \end{aligned} \tag{110}$$

In fact, we have to consider a weak form of (110), which is given by the following proposition:

Proposition 4. We define the differential operator Θ_t^ε as

$$\Theta_t^\varepsilon \varphi = \int_0^{\bar{\tau}} d\sigma\, (\sigma\nabla_x \cdot + \nabla_v \cdot)\, \mathbb{E}\big[E^\varepsilon(t-\varepsilon^2\sigma, x-\sigma v) \otimes E^\varepsilon(t,x)\big]\nabla_v\varphi, \quad \varphi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^d), \tag{111}$$

and the bilinear form ν_t^ε as

$$\nu_t^\varepsilon(\psi,\varphi) = \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \fint dx\, \psi\Theta_t^\varepsilon\varphi, \quad \forall \psi \in L^\infty(\mathbb{R}^+ \times Q), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^d). \tag{112}$$

Then, the weak formulation of (110) reads $\forall \varphi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^d)$,

$$\begin{aligned} &\int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv\, \nabla_v\varphi \cdot \mathbb{E}\Big[\fint dx\, \frac{f^\varepsilon(t)E^\varepsilon(t)}{\varepsilon}\Big] \\ &\qquad = \nu_t^\varepsilon\big(\mathbb{E}[f^\varepsilon(t)],\varphi\big) + \nu_t^\varepsilon\big(\mathbb{E}\big[S_{2\varepsilon^2\bar{\tau}}^\varepsilon f^\varepsilon(t-2\varepsilon^2\bar{\tau}) - f^\varepsilon(t)\big],\varphi\big) + \mu_t^\varepsilon(\varphi), \end{aligned} \tag{113}$$

where the remainder term $\mu_t^\varepsilon(\varphi)$ is given by

$$\begin{aligned} \mu_t^\varepsilon(\varphi) = -\varepsilon^4 \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \int_0^{\bar{\tau}} d\sigma \int_0^{2\bar{\tau}-\sigma} ds \fint dx \\ \mathbb{E}\big[f^\varepsilon(t-\varepsilon^2(\sigma+s))\, \mathcal{N}_{t-\varepsilon^2(\sigma+s)}^\varepsilon S_{-\varepsilon^2 s}^\varepsilon \mathcal{N}_{t-\varepsilon^2\sigma}^\varepsilon S_{-\varepsilon^2\sigma}^\varepsilon \mathcal{N}_t^\varepsilon\varphi\big]. \end{aligned} \tag{114}$$

Proof. We have to show that the weak formulation of (110) is given by (113). The left-hand side of (113) is obtained straightforwardly from the left-hand side of (110). The first two terms of the right-hand side of (113) can be obtained in a similar way from the first two terms of the right-hand side of (110), respectively. Indeed, we consider a non-random function $\psi = \psi(t,x,v)$, which can be either $\mathbb{E}[f^\varepsilon(t)]$ or $\mathbb{E}\big[S_{2\varepsilon^2\bar{\tau}}^\varepsilon f^\varepsilon(t-2\varepsilon^2\bar{\tau}) - f^\varepsilon(t)\big]$. Then, the quantity of interest is

$$\int_0^{\varepsilon^2\bar{\tau}} d\sigma \fint dx\, \mathbb{E}\big[\mathcal{N}_t^\varepsilon S_\sigma^\varepsilon \mathcal{N}_{t-\sigma}^\varepsilon S_{-\sigma}^\varepsilon\big]\psi. \tag{115}$$

Multiplying (115) by a test function $\varphi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^d)$ and integrating with respect to the time and velocity variables, we obtain, after expanding all operators,

$$\begin{aligned} \frac{1}{\varepsilon^2} \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv\, \varphi(t,v) \fint dx \int_0^{\varepsilon^2\bar{\tau}} d\sigma \\ \nabla_v \cdot \Big(\mathbb{E}\big[E^\varepsilon(t,x) \otimes E^\varepsilon(t-\sigma, x-\sigma v/\varepsilon^2)\big]\Big(\frac{\sigma}{\varepsilon^2}\nabla_x + \nabla_v\Big)\psi(t,x,v)\Big). \end{aligned} \tag{116}$$

Using integrations by parts with respect to the variables x and v and making the change of time variable $\sigma' = \sigma/\varepsilon^2$, we obtain from (116),

$$\int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \fint dx\, \psi(t,x,v) \int_0^{\bar\tau} d\sigma (\sigma\nabla_x\cdot + \nabla_v\cdot)\big(\mathbb{E}\big[E^\varepsilon(t-\varepsilon^2\sigma, x-\sigma v)\otimes E^\varepsilon(t,x)\big]\nabla_v\varphi(t,v)\big)$$
$$= \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \fint dx\, \psi\Theta_t^\varepsilon\varphi = \nu_t^\varepsilon(\psi,\varphi). \tag{117}$$

The first two terms of the right-hand side of (113) follow by replacing, respectively, ψ by $\mathbb{E}[f^\varepsilon(t)]$ and $\mathbb{E}\big[S^\varepsilon_{2\varepsilon^2\bar\tau}f^\varepsilon(t-2\varepsilon^2\bar\tau) - f^\varepsilon(t)\big]$ in (117). To obtain the weak formulation of the remainder term μ_t^ε, we use the skew-adjointness of $\mathcal{N}_t^\varepsilon$ and the dual of S_t^ε given by $S_t^{\varepsilon\,*} = S^\varepsilon_{-t}$. Using multiple velocity integrations by parts, several changes of variables in space and the changes of time variables $\sigma' = \sigma/\varepsilon^2$ and $s' = s/\varepsilon^2$, we obtain from the third term of the right-hand side of (113),

$$\begin{aligned}
\mu_t^\varepsilon(\varphi) &= \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv\, \varphi\mu_t^\varepsilon \\
&= \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv\, \varphi \int_0^{\varepsilon^2\bar\tau} d\sigma \int_0^{2\varepsilon^2\bar\tau-\sigma} ds \fint dx\, \mathbb{E}\big[\mathcal{N}_t^\varepsilon S_\sigma^\varepsilon \mathcal{N}^\varepsilon_{t-\sigma} S_s^\varepsilon \mathcal{N}^\varepsilon_{t-\sigma-s} f^\varepsilon(t-\sigma-s)\big] \\
&= \int_{\mathbb{R}^+} dt \int_0^{\varepsilon^2\bar\tau} d\sigma \int_0^{2\varepsilon^2\bar\tau-\sigma} ds\, \mathbb{E}\Big[\int_{\mathbb{R}^d} dv \fint dx\, \varphi \mathcal{N}_t^\varepsilon S_\sigma^\varepsilon \mathcal{N}^\varepsilon_{t-\sigma} S_s^\varepsilon \mathcal{N}^\varepsilon_{t-\sigma-s} f^\varepsilon(t-\sigma-s)\Big] \\
&= -\int_0^{\varepsilon^2\bar\tau} d\sigma \int_0^{2\varepsilon^2\bar\tau-\sigma} ds\, \mathbb{E}\Big[\int_{\mathbb{R}^d} dv \fint dx\, f^\varepsilon(t-\sigma-s)\mathcal{N}^\varepsilon_{t-\sigma-s} S^\varepsilon_{-s} \mathcal{N}^\varepsilon_{t-\sigma} S^\varepsilon_{-\sigma}\mathcal{N}_t^\varepsilon\varphi\Big] \\
&= -\varepsilon^4 \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \int_0^{\bar\tau} d\sigma \int_0^{2\bar\tau-\sigma} ds \fint dx \\
&\qquad \mathbb{E}\big[f^\varepsilon(t-\varepsilon^2(\sigma+s))\mathcal{N}^\varepsilon_{t-\varepsilon^2(\sigma+s)} S^\varepsilon_{-\varepsilon^2 s}\mathcal{N}^\varepsilon_{t-\varepsilon^2\sigma} S^\varepsilon_{-\varepsilon^2\sigma}\mathcal{N}_t^\varepsilon\varphi\big],
\end{aligned}$$

which is (114). This ends the Proof of Proposition 4. □

The next lemma states the limit of the operator Θ_t^ε as $\varepsilon \to 0$.

Lemma 4. Under hypothesis (H3), *for all* $\varphi \in \mathcal{D}(\mathbb{R}^+\times\mathbb{R}^d)$, *the operator* Θ_t^ε *defined by (111) becomes*

$$\Theta_t^\varepsilon\varphi = \nabla_v\cdot\Big(\Big(\int_0^{\bar\tau} d\sigma\, \mathcal{R}_{\bar\tau}(t-\varepsilon^2\sigma, t, -\sigma, -\sigma v)\Big)\nabla_v\varphi\Big), \tag{118}$$

and we obtain

$$\Theta_t^\varepsilon\varphi \to \Theta_t^0\varphi \quad in\ L^1(\mathbb{R}^+\times\mathbb{R}^d), \tag{119}$$

where the operator Θ_t^0 *is defined by*

$$\Theta_t^0\varphi = \nabla_v\cdot\Big(\Big(\int_0^{\bar\tau} d\sigma\, \mathcal{R}_{\bar\tau}(t, t, -\sigma, -\sigma v)\Big)\nabla_v\varphi\Big). \tag{120}$$

Proof. Using assumption (H3), i.e., $\mathbb{E}[E(t,\tau,x)\otimes E(s,\sigma,y)] = \mathcal{R}_{\bar\tau}(t,s,\tau-\sigma,x-y)$, for all $\varphi\in\mathcal{D}(\mathbb{R}^+\times\mathbb{R}^d)$, we obtain from (111),

$$\begin{aligned}
\Theta_t^\varepsilon\varphi &= \int_0^{\bar\tau} d\sigma\, (\sigma\nabla_x\cdot + \nabla_v\cdot)\, \mathbb{E}\big[E^\varepsilon(t-\varepsilon^2\sigma, x-\sigma v)\otimes E^\varepsilon(t,x)\big]\nabla_v\varphi \\
&= \int_0^{\bar\tau} d\sigma\, (\sigma\nabla_x\cdot + \nabla_v\cdot)\mathcal{R}_{\bar\tau}(t-\varepsilon^2\sigma, t, -\sigma, -\sigma v)\nabla_v\varphi \\
&= \int_0^{\bar\tau} d\sigma\, \nabla_v\cdot\big(\mathcal{R}_{\bar\tau}(t-\varepsilon^2\sigma, t, -\sigma, -\sigma v)\nabla_v\varphi\big) \\
&= -\int_0^{\bar\tau} d\sigma\, \sigma(\nabla_x\cdot\mathcal{R}_{\bar\tau})(t-\varepsilon^2\sigma, t, -\sigma, -\sigma v)\cdot\nabla_v\varphi + \int_0^{\bar\tau} d\sigma\, \mathcal{R}_{\bar\tau}(t-\varepsilon^2\sigma, t, -\sigma, -\sigma v) : \nabla_v^2\varphi.
\end{aligned}$$

Since $\mathcal{R}_{\bar\tau}\in L^\infty(\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}; W^{2,\infty}(\mathbb{T}^d))$ and because $\bar\tau$ is bounded, we obtain

$$\int_0^{\bar\tau} d\sigma\, \sigma(\nabla_x\cdot\mathcal{R}_{\bar\tau})(t-\varepsilon^2\sigma, t, -\sigma, -\sigma v)\cdot\nabla_v\varphi \to \int_0^{\bar\tau} d\sigma\, \sigma(\nabla_x\cdot\mathcal{R}_{\bar\tau})(t, t, -\sigma, -\sigma v)\cdot\nabla_v\varphi,$$

for almost every $(t, v) \in \mathbb{R}^+ \times \mathbb{R}^d$ as $\varepsilon \to 0$ and

$$\int_0^{\bar{\tau}} d\sigma\, \mathcal{R}_{\bar{\tau}}(t - \varepsilon^2\sigma, t, -\sigma, -\sigma v) : \nabla_v^2\varphi \to \int_0^{\bar{\tau}} d\sigma\, \mathcal{R}_{\bar{\tau}}(t, t, -\sigma, -\sigma v) : \nabla_v^2\varphi,$$

for almost every $(t, v) \in \mathbb{R}^+ \times \mathbb{R}^d$ as $\varepsilon \to 0$. Moreover, using the regularity of $\mathcal{R}_{\bar{\tau}}$, we obtain

$$|\Theta_t^\varepsilon \varphi| \le \|\sigma\nabla_x\mathcal{R}_{\bar{\tau}}\|_{L^\infty(\mathbb{R}_t^+\times\mathbb{R}_t^+;L^1([-\bar{\tau},\bar{\tau}]_\sigma;L^\infty(\mathbb{T}^d)))}|\nabla_v\varphi| + \|\mathcal{R}_{\bar{\tau}}\|_{L^\infty(\mathbb{R}_t^+\times\mathbb{R}_t^+;L^1([-\bar{\tau},\bar{\tau}]_\sigma;L^\infty(\mathbb{T}^d)))}|\nabla_v^2\varphi|, \tag{121}$$

where the right-hand side of (121) defines a function in $L^1(\mathbb{R}^+ \times \mathbb{R}^d)$ independent of ε. Therefore, using the Lebesgue dominated convergence theorem, we obtain the limit (119), where the limit point is given by (120). This ends the proof. □

To deal with the second term of right-hand side of (113), we use the following lemma:

Lemma 5. Under hypothesis (H3), *for all* $\varphi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^d)$, *we obtain*

$$\lim_{\varepsilon\to 0} \nu_t^\varepsilon\big(\mathbb{E}[S_{2\varepsilon^2\bar{\tau}}^\varepsilon f^\varepsilon(t - 2\varepsilon^2\bar{\tau}) - f^\varepsilon(t)], \varphi\big) = 0.$$

Proof. Using definitions (112) and (118) of, respectively, the bilinear form ν_t^ε and the operator Θ_t^ε, we obtain

$$\begin{aligned}\nu_t^\varepsilon\big(\mathbb{E}[S_{2\varepsilon^2\bar{\tau}}^\varepsilon f^\varepsilon(t - 2\varepsilon^2\bar{\tau}) - f^\varepsilon(t)], \varphi\big) &= \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \fint dx\, \mathbb{E}[S_{2\varepsilon^2\bar{\tau}}^\varepsilon f^\varepsilon(t - 2\varepsilon^2\bar{\tau}) - f^\varepsilon(t)]\Theta_t^\varepsilon\varphi \\ &= \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv\, \Theta_t^\varepsilon\varphi\; \mathbb{E}\Big[\fint dx\, f^\varepsilon(t - 2\varepsilon^2\bar{\tau}) - \fint dx\, f^\varepsilon(t)\Big].\end{aligned} \tag{122}$$

On the one hand, from Proposition 3, we obtain $\mathbb{E}[\fint dx\, f^\varepsilon] = \fint dx\, \mathbb{E}[f^\varepsilon] \rightharpoonup \fint dx\, f = f$ in $L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ weak-$*$ as $\varepsilon \to 0$. Then, $\mathbb{E}[\fint dx\, f^\varepsilon(t - 2\varepsilon^2\bar{\tau}) - \fint dx\, f^\varepsilon(t)] \rightharpoonup 0$ in $L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ weak-$*$ as $\varepsilon \to 0$. On the other hand, from Lemma 4, we obtain $\Theta_t^\varepsilon\varphi \to \Theta_t^0\varphi$ in $L^1(\mathbb{R}^+ \times \mathbb{R}^d)$ strong as $\varepsilon \to 0$. Therefore, we can pass to the limit $\varepsilon \to 0$ in (122), and we obtain the desired result. □

The asymptotic behavior of the term μ_t^ε is given by the following lemma:

Lemma 6. Under hypothesis (H4), *for all* $\varphi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^d)$, *we obtain*

$$|\mu_t^\varepsilon(\varphi)| \le \varepsilon\bar{\tau}^4 C_0 C_E \|\varphi\|_{L^\infty(\mathbb{R}^+;W^{3,1}(\mathbb{R}^d))}. \tag{123}$$

Proof. Introducing the operator $\Gamma_{t,\sigma}^\varepsilon$, defined by

$$\Gamma_{t,\sigma}^\varepsilon = \mathcal{N}_{t-\varepsilon^2\sigma}^\varepsilon S_{-\varepsilon^2\sigma}^\varepsilon, \tag{124}$$

we obtain from definition (114) of $\mu_t^\varepsilon(\varphi)$,

$$\begin{aligned}|\mu_t^\varepsilon(\varphi)| &\lesssim \varepsilon^4\bar{\tau}^2\|f^\varepsilon(t - \varepsilon^2(\sigma + s))\|_{L^\infty(\mathbb{R}_t^+\times[0,\bar{\tau}]_\sigma\times[0,2\bar{\tau}]_s\times Q)} \\ &\quad \mathbb{E}\big[\|\Gamma_{t-\varepsilon^2\sigma,s}^\varepsilon\Gamma_{t,\sigma}^\varepsilon\mathcal{N}_t^\varepsilon\varphi\|_{L^\infty(\mathbb{R}_t^+\times[0,\bar{\tau}]_\sigma\times[0,2\bar{\tau}]_s;L^1(Q))}\big] \\ &\lesssim C_0\varepsilon^4\bar{\tau}^2\mathbb{E}\big[\|\Gamma_{t-\varepsilon^2\sigma,s}^\varepsilon\Gamma_{t,\sigma}^\varepsilon\mathcal{N}_t^\varepsilon\varphi\|_{L^\infty(\mathbb{R}_t^+\times[0,\bar{\tau}]_\sigma\times[0,2\bar{\tau}]_s;L^1(Q))}\big].\end{aligned} \tag{125}$$

Introducing the translation operator $T_{-\sigma v}$ in the x-direction, defined by

$$T_{-\sigma v}\psi(t, x, v) = \psi(t, x + \sigma v, v),$$

we obtain for any smooth function $\psi = \psi(t, x, v)$,

$$\Gamma^{\varepsilon}_{t,\sigma}\psi = -\varepsilon^{-1}E^{\varepsilon}(t-\varepsilon^2\sigma,x)\cdot\nabla_v(T_{-\sigma v}\psi)$$
$$= -\varepsilon^{-1}E(t-\varepsilon^2\sigma,t/\varepsilon^2-\sigma,x)\cdot(\sigma T_{-\sigma v}\nabla_x\psi + T_{-\sigma v}\nabla_v\psi). \tag{126}$$

Using (124) and (126), we obtain for any smooth function $\psi = \psi(t,x,v)$,

$$\Gamma^{\varepsilon}_{t-\varepsilon^2\sigma,s}\Gamma^{\varepsilon}_{t,\sigma}\psi = \mathcal{N}^{\varepsilon}_{t-\varepsilon^2(\sigma+s)}S^{\varepsilon}_{-\varepsilon^2 s}\Gamma^{\varepsilon}_{t,\sigma}\psi$$
$$= -\varepsilon^{-1}\mathcal{N}^{\varepsilon}_{t-\varepsilon^2(\sigma+s)}\big(T_{-sv}E^{\varepsilon}(t-\varepsilon^2\sigma,x)\big)\cdot T_{-(\sigma+s)v}(\sigma\nabla_x\psi+\nabla_v\psi)$$
$$= \varepsilon^{-2}\sum_{i,j}E^{\varepsilon}_i(t-\varepsilon^2(\sigma+s),x)\big\{sT_{-sv}\partial_{x_i}E^{\varepsilon}_j(t-\varepsilon^2\sigma,x)\big[\sigma T_{-(\sigma+s)v}\partial_{x_j}\psi$$
$$+\ T_{-(\sigma+s)v}\partial_{v_j}\psi\big]\ +\ T_{-sv}E^{\varepsilon}_j(t-\varepsilon^2\sigma,x)\big[\sigma(\sigma+s)T_{-(\sigma+s)v}\partial^2_{x_ix_j}\psi$$
$$+\ \sigma T_{-(\sigma+s)v}\partial^2_{v_ix_j}\psi\ +\ (\sigma+s)T_{-(\sigma+s)v}\partial^2_{x_iv_j}\psi\ +\ T_{-(\sigma+s)v}\partial^2_{v_iv_j}\psi\big]\big\}. \tag{127}$$

Using (127), we obtain for all $\psi \in L^{\infty}(\mathbb{R}^+;W^{2,1}(Q))$,

$$\|\Gamma^{\varepsilon}_{t-\varepsilon^2\sigma,s}\Gamma^{\varepsilon}_{t,\sigma}\psi\|_{L^{\infty}(\mathbb{R}^+_t\times[0,\bar{\tau}]_{\sigma}\times[0,2\bar{\tau}]_s;L^1(Q))} \lesssim \frac{\bar{\tau}^2}{\varepsilon^2}\|E\|^2_{L^{\infty}(\mathbb{R}^+\times\mathbb{R}^+;W^{1,\infty}(\mathbb{T}^d))}\|\psi\|_{L^{\infty}(\mathbb{R}^+;W^{2,1}(Q))}. \tag{128}$$

Using (128), we obtain for all $\varphi \in L^{\infty}(\mathbb{R}^+;W^{3,1}(\mathbb{R}^d))$,

$$\|\Gamma^{\varepsilon}_{t-\varepsilon^2\sigma,s}\Gamma^{\varepsilon}_{t,\sigma}\mathcal{N}^{\varepsilon}_t\varphi\|_{L^{\infty}(\mathbb{R}^+_t\times[0,\bar{\tau}]_{\sigma}\times[0,2\bar{\tau}]_s;L^1(Q))} \lesssim \frac{\bar{\tau}^2}{\varepsilon^3}\|E\|^3_{L^{\infty}(\mathbb{R}^+\times\mathbb{R}^+;W^{2,\infty}(\mathbb{T}^d))}\|\varphi\|_{L^{\infty}(\mathbb{R}^+;W^{3,1}(\mathbb{R}^d))}. \tag{129}$$

Combining (125) and (129), we obtain from hypothesis (H4) the final estimate (123), which ends the Proof of Lemma 6.

We are now able to conclude the Proof of Theorem 2 by showing that we can pass to the limit $\varepsilon \to 0$ in (97). From (97) and (113), we have for all $\varphi \in \mathcal{D}(\mathbb{R}^+\times\mathbb{R}^d)$,

$$\int_{\mathbb{R}^+}dt\int_{\mathbb{R}^d}dv\,\mathbb{E}\Big[⨍dx\,f^{\varepsilon}\Big]\partial_t\varphi - \int_{\mathbb{R}^+}dt\int_{\mathbb{R}^d}dv\,\nabla_v\varphi\cdot\mathbb{E}\Big[⨍dx\,\frac{f^{\varepsilon}E^{\varepsilon}}{\varepsilon}\Big] = 0, \tag{130}$$

with

$$\int_{\mathbb{R}^+}dt\int_{\mathbb{R}^d}dv\,\nabla_v\varphi\cdot\mathbb{E}\Big[⨍dx\,\frac{f^{\varepsilon}E^{\varepsilon}}{\varepsilon}\Big]$$
$$= \nu^{\varepsilon}_t\big(\mathbb{E}[f^{\varepsilon}],\varphi\big) + \nu^{\varepsilon}_t\big(\mathbb{E}\big[S^{\varepsilon}_{2\varepsilon^2\bar{\tau}}f^{\varepsilon}(t-2\varepsilon^2\bar{\tau})-f^{\varepsilon}(t)\big],\varphi\big) + \mu^{\varepsilon}_t(\varphi). \tag{131}$$

From Proposition 3, we have $\mathbb{E}[f^{\varepsilon}] \rightharpoonup f$ in $L^{\infty}(\mathbb{R}^+\times Q)$ weak−$*$ as $\varepsilon\to 0$. Then, we obtain $\mathbb{E}[⨍dx\,f^{\varepsilon}] = ⨍dx\,\mathbb{E}[f^{\varepsilon}] \rightharpoonup ⨍dx\,f = f$ in $L^{\infty}(\mathbb{R}^+\times\mathbb{R}^d)$ weak−$*$ as $\varepsilon\to 0$. Using this weak limit, we obtain for the first term of (130),

$$\int_{\mathbb{R}^+}dt\int_{\mathbb{R}^d}dv\,\mathbb{E}\Big[⨍dx\,f^{\varepsilon}\Big]\partial_t\varphi \longrightarrow \int_{\mathbb{R}^+}dt\int_{\mathbb{R}^d}dv\,f\partial_t\varphi \quad \text{as } \varepsilon\to 0. \tag{132}$$

Using definitions (112) and (118) of, respectively, the bilinear form ν^{ε}_t and the operator Θ^{ε}_t, we obtain

$$\nu^{\varepsilon}_t\big(\mathbb{E}[f^{\varepsilon}],\varphi\big) = \int_{\mathbb{R}^+}dt\int_{\mathbb{R}^d}dv⨍dx\,\mathbb{E}[f^{\varepsilon}]\Theta^{\varepsilon}_t\varphi$$
$$= \int_{\mathbb{R}^+}dt\int_{\mathbb{R}^d}dv\,\mathbb{E}\Big[⨍dx\,f^{\varepsilon}\Big]\Theta^{\varepsilon}_t\varphi. \tag{133}$$

Moreover, from Lemma 4, we obtain $\Theta^{\varepsilon}_t\varphi \to \Theta^0_t\varphi$ in $L^1(\mathbb{R}^+\times\mathbb{R}^d)$ strong as $\varepsilon\to 0$. Using the weak limit $\mathbb{E}[⨍dx\,f^{\varepsilon}] \rightharpoonup f$ in $L^{\infty}(\mathbb{R}^+\times\mathbb{R}^d)$ weak−$*$ as $\varepsilon\to 0$, we can then pass to the limit in (133) or in the first term of the right-hand side of (131), and we obtain

$$\nu_t^\varepsilon(\mathbb{E}[f^\varepsilon],\varphi) \to \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv\, f\Theta_t^0\varphi \quad \text{as } \varepsilon \to 0. \tag{134}$$

Using Lemma 5, the second term of the right-hand side of (131) vanishes as $\varepsilon \to 0$. Finally, for the third term of the right-hand side of (131), we obtain from Lemma 6 that $\mu_t^\varepsilon \rightharpoonup 0$ in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d)$ as $\varepsilon \to 0$. Therefore, we obtain from (131) and (134),

$$\int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv\, \nabla_v\varphi \cdot \mathbb{E}\Big[⨍ dx\, \frac{f^\varepsilon E^\varepsilon}{\varepsilon}\Big] \to \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv\, f\Theta_t^0\varphi \quad \text{as } \varepsilon \to 0, \tag{135}$$

where

$$\int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv\, f\Theta_t^0\varphi = \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv\, \varphi\nabla_v \cdot \Big(\Big(\int_0^{\bar\tau} d\sigma\, \mathcal{R}_{\bar\tau}(t,t,\sigma,\sigma v)\Big)\nabla_v f\Big). \tag{136}$$

Passing to the limit $\varepsilon \to 0$ in (130) and using (132)–(136), we obtain

$$\partial_t f - \nabla_v \cdot \Big(\Big(\int_0^{\bar\tau} d\sigma\, \mathcal{R}_{\bar\tau}(t,t,\sigma,\sigma v)\Big)\nabla_v f\Big) = 0 \quad in\ \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d),$$

which shows (84) with (83).

It remains to show time continuity of the limit point f. From (97) and the convergence result (135), we have for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$,

$$\frac{d}{dt}\int_{\mathbb{R}^d} dv\, \varphi\mathbb{E}\Big[⨍ dx\, f^\varepsilon\Big] = \int_{\mathbb{R}^d} dv\, \nabla_v\varphi \cdot \mathbb{E}\Big[⨍ dx\, \frac{f^\varepsilon E^\varepsilon}{\varepsilon}\Big] \le C(\varphi), \quad \forall t \in \mathbb{R}^+,$$

where the constant $C(\varphi) > 0$ depends on φ but not on t and ε. Therefore, we obtain

$$\begin{aligned}\Big|\int_{\mathbb{R}^d} dv \Big(\mathbb{E}\Big[⨍ dx\, f^\varepsilon(t)\Big] - \mathbb{E}\Big[⨍ dx\, f^\varepsilon(t-2\varepsilon^2\bar\tau)\Big]\Big)\varphi\Big| &\le \Big|\int_{t-2\varepsilon^2\bar\tau}^t ds \int_{\mathbb{R}^d} dv ⨍ dx\, \varepsilon^{-1}\mathbb{E}[E^\varepsilon(s,x)\cdot\nabla_v f^\varepsilon(s,x,v)]\varphi(v)\Big| \\ &\le \varepsilon^{-1}\|f_0^\varepsilon\|_{L^\infty(Q)} \int_{t-2\varepsilon^2\bar\tau}^t ds \int_{\mathbb{R}^d} dv ⨍ dx\, |\nabla_v\varphi(v)|\mathbb{E}[|E^\varepsilon(s,x)|] \\ &\le 2\varepsilon\bar\tau C_0 C_E \|\varphi\|_{W^{1,1}(\mathbb{R}^d)} \longrightarrow 0 \quad \text{as } \varepsilon \to 0.\end{aligned}$$

Therefore, the set $\{\int_{\mathbb{R}^d} dv\, \mathbb{E}[⨍ dx\, f^\varepsilon(t)]\varphi\}_{\varepsilon>0}$ is uniformly equi-continuous in time, and by Ascoli–Arzela theorem, this set is relatively compact in $\mathscr{C}([0,T])$. Then, there exists a subsequence, still labeled by ε, such that $\int_{\mathbb{R}^d} dv\, \mathbb{E}[⨍ dx\, f^\varepsilon(t)]\varphi$ converges uniformly in time to $\int_{\mathbb{R}^d} f(t)\varphi dv$, with, in particular, $f_{|t=0} = ⨍ dx\, f_0$. Finally, the bound

$$\begin{aligned}\Big\|\mathbb{E}\Big[⨍ dx\, f^\varepsilon\Big]\Big\|_{L^\infty(0,T;L^p(\mathbb{R}^d))} &\le (2\pi)^{-d/p}\,\mathbb{E}\big[\|f^\varepsilon\|_{L^\infty(0,T;L^p(Q))}\big] \\ &= (2\pi)^{-d/p}\,\mathbb{E}\big[\|f_0^\varepsilon\|_{L^p(Q)}\big] \le (2\pi)^{-d/p}C_0 < \infty\end{aligned}$$

(obtained by using Hölder's inequality) and the density of $\mathcal{D}(\mathbb{R}^d)$ in $L^q(\mathbb{R}^d)$ with $1/p + 1/q = 1$ and $1 < q < \infty$ imply by standard arguments that $\mathbb{E}[⨍ dx\, f^\varepsilon(t)] \rightharpoonup f$ in $\mathscr{C}([0,T]; L^p(\mathbb{R}^d) - weak)$ for all $T > 0$ and $1 < p < \infty$. This completes the Proof of Theorem 2.

D. Links with some kinetic turbulence theories of plasma physics

In this section, we relate the results of Sec. IV B to two kinetic turbulence theories of plasma physics. First, when the autocorrelation time of particles $\bar{\tau}$ is finite, our result leads to a diffusion matrix, which is reminiscent of the diffusion matrix of the resonance broadening theory,[1,21,27,51,58] a refinement of the quasilinear theory. This diffusion matrix falls into the framework of Theorem 2. Second, even if Theorem 2 and, in particular, hypothesis (H2) hold only for $\bar{\tau}$ finite, it is worthwhile to pass, at least formally, to the limit $\bar{\tau} \to +\infty$. When the autocorrelation time of particles $\bar{\tau}$ tends to infinity, we recover formally the structure of the diffusion matrix of the quasilinear theory[26,57] from the diffusion matrix (83).

1. Resonance broadening like theory: Finite $\bar{\tau}$

Here, we show how to relate the diffusion matrix (83) to the diffusion matrix obtained in the resonance broadening theory.[1,21,27,51,58] The diffusion matrix (83), obtained in Theorem 2, can be recast as

$$\mathscr{D}_{\bar{\tau}}(t,v) = \int_0^{\bar{\tau}} d\sigma\, \mathbb{E}[E(t,0,x) \otimes E(t,-\sigma,x-\sigma v)]. \tag{137}$$

Introducing the Fourier series decomposition of E,

$$E(t,\tau,x) = \sum_{k\in\mathbb{Z}^d} e^{\mathrm{i}k\cdot x}\hat{E}(t,\tau,k),$$

we can suppose without loss of generality that $\hat{E}(t,\tau,k) = e^{-\mathrm{i}\omega(k)\tau}\tilde{E}(t,\tau,k)$ where the real-valued function $\mathbb{Z}^d \ni k \mapsto \omega(k) \in \mathbb{R}$ is odd, i.e., $\omega(-k) = -\omega(k)$ for all $k \in \mathbb{Z}^d$. This transformation, which can be seen as a WKB ansatz, is just a change of unknown functions. In dimensional variables, $\omega(k)$ should scale as ω_p, which means that τ (respectively, t) represents the fast (respectively, slow) time variable. In the framework of the resonance broadening theory of plasma physics, which is self-consistent, frequencies $\omega(k)$ are given by the resolution of the dispersion relation (4) or by its approximation (5). Since here we work in a non-self-consistent frame, we suppose that frequencies $\omega(k)$ are simply given by a suitable function of k, which is regular and bounded with respect to k. In the same spirit as assumption (H3), we now make the following assumption:

(H3′) There exist a non-negative real-valued function $\mathcal{E}(t,k) : \mathbb{R}^+ \times \mathbb{Z}^d \to \mathbb{R}^+$, with $\mathcal{E}(t,k) = \mathcal{E}(t,-k)$ and $|k|^2|\mathcal{E}(t,k)|^{1/2} \in L^\infty(\mathbb{R}^+;\ell^1(\mathbb{Z}^d))$, and a bounded function $A_{\bar{\tau}}(\tau,k) : [-\bar{\tau},\bar{\tau}] \times \mathbb{Z}^d \to \mathbb{R}^+$, even and compactly supported in τ, such that

$$\mathbb{E}\big[\tilde{E}(t,\tau,k) \otimes \tilde{E}(t,\sigma,k')\big] = A_{\bar{\tau}}(\tau-\sigma,k)\mathcal{E}(t,k)\,\frac{k\otimes k}{|k|^2}\,\delta(k+k').$$

The term $\delta(k+k')$, which provides spatial homogeneity, is reminiscent of what plasma physicists call the random phase approximation. Indeed, the random phase approximation assumes that $\tilde{E}(t,\tau,k) \propto \exp(\mathrm{i}\phi_k)$, where $(\phi_k)_{k\in\mathbb{Z}^d}$ are independent random variables equidistributed on $[0,2\pi]$ such that $\phi_{-k} = -\phi_k$. The matrix $k\otimes k/|k|^2$ means that we choose an electric field which is the gradient of an electric potential. The function $A_{\bar{\tau}}$ corresponds to a time autocorrelation function. We suppose that the function $\tau \mapsto A_{\bar{\tau}}(\tau,k)$ is bounded, even, and with compact support included in $[-\bar{\tau},\bar{\tau}]$ for all $k \in \mathbb{Z}^d$. The function $\mathcal{E}(t,k)$ corresponds to the energy of the kth mode of the electric field, whose time scale of evolution in dimensional variables is τ_L, i.e., a slow time scale in comparison to $1/\omega_p$ [remember that $1/(\omega_p\tau_L) = \varepsilon^2$]. In the self-consistent framework of resonance broadening theory, the function $\mathcal{E}(t,k)$ are given by Eq. (2) in which the energy of the kth mode of the electric field, $|E(t,k)|^2$, is replaced by $\mathcal{E}(t,k)$ and the grow rate $\gamma(t,k)$ is given by a d-dimensional version of (4) [or its approximation (6)]. Since here we work in a non-self-consistent frame, we also suppose that energy amplitudes $\mathcal{E}(t,k)$ are simply given by a suitable bounded function, which decreases fast enough in the variable k to satisfy hypothesis (H4). Hypothesis (H3′), which is a particular case of hypothesis (H3), is consistent with the spatiotemporal homogeneity property of the turbulence. Actually, the property (H3′) implies the property (H3), in other words, the property (H3′) is less general than the property (H3). Indeed, we obtain from (H3′),

$$\mathbb{E}[E(t,\tau,x)\otimes E(t,\sigma,y)] = \sum_{k\in\mathbb{Z}^d} A_{\bar{\tau}}(\tau-\sigma,k)\mathcal{E}(t,k)\frac{k\otimes k}{|k|^2}e^{\mathrm{i}k\cdot(x-y)}e^{-\mathrm{i}\omega(k)(\tau-\sigma)},$$

from which we easily observe that the spatiotemporal autocorrelation function $\mathbb{E}[E(t,\tau,x)\otimes E(t,\sigma,y)]$ is invariant under time and space translations. In terms of the autocorrelation matrix $\mathcal{R}_{\bar{\tau}}$, hypothesis (H3′) is equivalent to

$$\mathcal{R}_{\bar{\tau}}(t,t,\sigma,x) = \sum_{k\in\mathbb{Z}^d} e^{-\mathrm{i}\omega(k)\sigma}A_{\bar{\tau}}(\sigma,k)\mathcal{E}(t,k)\frac{k\otimes k}{|k|^2}e^{\mathrm{i}k\cdot x}. \tag{138}$$

Using assumption (H3′) in (137), where the electric field E is written in terms of its Fourier series decomposition, we obtain

$$\mathscr{D}_{\bar{\tau}}(t,v) = \sum_{k\in\mathbb{Z}^d} \mathcal{E}(t,k)\frac{k\otimes k}{|k|^2}\int_0^{\bar{\tau}} d\sigma\, e^{-\mathrm{i}(\omega(k)-k\cdot v)\sigma} A_{\bar{\tau}}(\sigma,k). \tag{139}$$

The diffusion matrix (139) can be rewritten as

$$\mathscr{D}_{\bar{\tau}}(t,v) = \sum_{k\in\mathbb{Z}^d} \mathcal{E}(t,k)\frac{k\otimes k}{|k|^2} R_{\bar{\tau}}(\omega(k)-k\cdot v,k), \tag{140}$$

where the resonance function $R_{\bar{\tau}}$ is given by

$$\begin{aligned} R_{\bar{\tau}}(\omega(k)-k\cdot v,k) &= \Re e\int_0^{\bar{\tau}} d\sigma\, e^{-\mathrm{i}(\omega(k)-k\cdot v)\sigma} A_{\bar{\tau}}(\sigma,k)\\ &= \frac{1}{2}\int_{-\infty}^{+\infty} d\sigma\, e^{-\mathrm{i}(\omega(k)-k\cdot v)\sigma} A_{\bar{\tau}}(\sigma,k). \end{aligned} \tag{141}$$

In (141), we have used the even parity and the compact support (included in $[-\bar{\tau},\bar{\tau}]$) of the real function $\tau\mapsto A_{\bar{\tau}}(\tau,k)$ to obtain the last equality.

Substituting hypothesis (H3′) for hypothesis (H3) in Theorem 2, we obtain the following corollary:

Corollary 2. Let E be an integrable random vector field satisfying assumptions (H1) *and* (H2) *and* (H3′) *and* (H4), *and let* E^ε *be given by (81). Let* $\{f_0^\varepsilon\}_{\varepsilon>0}$ *be a sequence of independent random non-negative initial data and* C_0 *be a positive constant such that for a.e.* $\varpi\in\mathcal{O}$, $\|f_0^\varepsilon\|_{L^1(Q)}+\|f_0^\varepsilon\|_{L^\infty(Q)}\le C_0<\infty$. *Let* $\mathscr{D}_{\bar{\tau}}$ *be the matrix-valued function defined by (140) and (141). Let* f^ε *be the unique weak solution of the Vlasov equation* (17), *with initial data* $f^\varepsilon_{|_{t=0}}=f_0^\varepsilon$. *Then, we have the following:*

1. *Up to extraction of a subsequence,* $\mathbb{E}[f_0^\varepsilon]$ *converges in* $L^\infty(Q)$ *weak−∗ to a function* $f_0\in L^1\cap L^\infty(Q)$, $\mathbb{E}[f^\varepsilon]$ *converges in* $L^\infty(\mathbb{R}^+;L^\infty(Q))$ *weak−∗ to a function* $f\in L^\infty(\mathbb{R}^+;L^1\cap L^\infty(\mathbb{R}^d))$, *and* $\mathbb{E}[⨍dx\, f^\varepsilon]$ *converges in* $L^\infty(\mathbb{R}^+;L^\infty(\mathbb{R}^d))$ *weak−∗ to f. Moreover,* $\mathbb{E}[⨍dx\, f^\varepsilon]$ *converges in* $\mathscr{C}(0,T;L^p(\mathbb{R}^d)-weak)$ *to f for* $1<p<\infty$ *and for all* $T>0$. *The limit point* $f=f(t,v)$ *is solution of the following diffusion equation in the sense of distributions:*

$$\begin{aligned} &\partial_t f-\nabla_v\cdot(\mathscr{D}_{\bar{\tau}}\nabla_v f)=0 \quad \text{in } \mathcal{D}'(\mathbb{R}^+\times\mathbb{R}^d),\\ &f_{|_{t=0}}=⨍dx\, f_0. \end{aligned}$$

2. *The diffusion matrix* $\mathscr{D}_{\bar{\tau}}$ *is symmetric, non-negative, and analytic in the velocity variables.*

Proof. The proof of point 1 of Corollary 2 is the same as the Proof of Theorem 2. It remains to deal with the proof of point 2. Symmetry of the diffusion matrix $\mathscr{D}_{\bar{\tau}}$ is obvious, while reality of the function $R_{\bar{\tau}}$ defined by (141) follows from the parity (even) of the function $\sigma\mapsto A_{\bar{\tau}}(\sigma,k)$. Non-negativeness of the diffusion matrix $\mathscr{D}_{\bar{\tau}}$ comes from the non-negativeness of the function $R_{\bar{\tau}}$, which results from both the Bochner theorem (see, e.g., Theorem 2 p. 346 in Ref. 60) and the fact that the function $\sigma\mapsto A_{\bar{\tau}}(\sigma,k)$ is positive definite in the following sense:

$$\int_{\mathbb{R}} d\tau\int_{\mathbb{R}} d\sigma A_{\bar{\tau}}(\tau-\sigma,k)\varphi(\tau)\varphi(\sigma)\ge 0 \tag{142}$$

for every continuous function φ with compact support. Indeed, from assumption (H3′) and using $\tilde{E}(t,\tau,-k)=\tilde{E}^*(t,\tau,k)$, for all vector $X\in\mathbb{R}^d\backslash\{0,k^\perp\}$ and $\varphi\in\mathscr{C}_c^0(\mathbb{R};\mathbb{R})$ (the set of continuous and compactly supported functions from $\mathbb{R}$ to $\mathbb{R}$), we obtain

$$\begin{aligned} &\frac{|k\cdot X|^2}{|k|^2}\mathcal{E}(t,k)\int_{\mathbb{R}} d\tau\int_{\mathbb{R}} d\sigma A_{\bar{\tau}}(\tau-\sigma,k)\varphi(\tau)\varphi(\sigma)\\ &\qquad = \mathbb{E}\left[\left(\int_{\mathbb{R}} d\tau\tilde{E}(t,\tau,k)\cdot X\varphi(\tau)\right)\left(\int_{\mathbb{R}} d\sigma\tilde{E}(t,\sigma,k)\cdot X\varphi(\sigma)\right)^*\right]\ge 0, \end{aligned}$$

which implies (142). Finally, $\mathscr{D}_{\bar{\tau}}(t,v)$, defined by (140) and (141), is an analytic function in v because the function $R_{\bar{\tau}}$ is the Fourier transform of the compactly supported function $A_{\bar{\tau}}$. □

The diffusion matrix (140) is reminiscent of the diffusion matrix of resonance broadening theory.[21,27,58] In Remark 13, we present very briefly the result of the resonance broadening theory to point out the similarities and the differences between the resonance broadening effect of Corollary 2 and that of the resonance broadening theory. As already mentioned in Sec. III B 2, from a mathematical point of view, the consequence of the finite-$\bar{\tau}$ or resonance broadening effect of Corollary 2 is a regularity improvement of the diffusion matrix in the velocity variables (by comparison with the diffusion matrix of the quasilinear theory, see Sec. IV D 2). From a physical point of view, our broadening resonance effect corresponds to a broadening in time frequency of a width $\Delta\omega = 1/\bar{\tau}$ for the resonance $\omega(k) - k \cdot v = 0$ ($\Delta\omega = 0$ for the quasilinear theory, see Sec. IV D 2). Here, up to hypothesis (H3′), we can choose freely the time autocorrelation function $A_{\bar{\tau}}$. From (141), we observe that the resonance function $R_{\bar{\tau}}$ is obtained as the Fourier transform of $A_{\bar{\tau}}/2$. In particular, we can recover diffusion matrices (80) and (87) obtained, respectively, in Sec. III B 2 and Corollary 1.

Remark 13 (resonance broadening theory). The finite-$\bar{\tau}$ or resonance broadening effect obtained in Corollary 2 is reminiscent of the resonance broadening theory[1,21,27,51,58], but it is not exactly the same since the derivation of resonance broadening theory is quite different and more involved. The resonance broadening theory is a correction to the quasilinear theory, which takes into account the particle diffusion coming from $\mathscr{D}_{\bar{\tau}}$ for calculating $\mathscr{D}_{\bar{\tau}}$ itself. Indeed, in the original derivation of the quasilinear theory, the diffusion matrix of the quasilinear is obtained by assuming that the perturbed dynamics of particles (called fluctuations) can be approximated by the ballistic motion or the free flow because fluctuations are small enough. This approximation is consistent with the limit $\bar{\tau} \to +\infty$ (or $\tau_D \to +\infty$ in dimensional variables) since for such limit, particles follow almost straight lines. If the fluctuation amplitude is sufficiently large and/or the wave spectrum is sufficiently narrow in k-space, the diffusion of particle trajectories can produce an appreciable broadening of wave–particle resonances $\omega(k) - k \cdot v \simeq 0$, even when $\mathcal{E}_{el}/\mathcal{E}_{kin} \ll 1$. To take these effects into account, the resonance broadening theory aims at modifying appropriately the diffusion matrix from the particle dynamics in a self-consistent way, i.e., by incorporating the diffusion of particle trajectories. Using a statistical approach,[21,27,58] the broadening resonance theory states that the autocorrelation function should be given by

$$\begin{aligned} &A_{\bar{\tau}}(\sigma,k) = \exp(-(\sigma/\bar{\tau})^3/3), \quad \text{with} \quad \bar{\tau} = (k\otimes k : \mathscr{D}_{\mathrm{rb}}(\sigma,v))^{-1/3}, \quad \text{i.e.,} \\ &A_{\bar{\tau}}(\sigma,k) = A_{\mathrm{rb}}(\sigma,k,\mathscr{D}_{\mathrm{rb}}(\sigma,v)) := \exp\left(-\frac{1}{3}k\otimes k : \mathscr{D}_{\mathrm{rb}}(\sigma,v)\sigma^3\right). \end{aligned} \tag{143}$$

We refer the reader to Remark 14 for a rough but short explanation of this kind of result in one dimension (see Refs. 27 and 58 for an original and detailed derivation). We now observe that the autocorrelation time $\bar{\tau}$ is no more a free parameter but is determined by the diffusion matrix $\mathscr{D}_{\mathrm{rb}}$. Likewise, the time autocorrelation function A_{rb} can no more be chosen freely but is an explicit function of the diffusion matrix $\mathscr{D}_{\mathrm{rb}}$. According to the resonance broadening theory, the diffusion matrix $\mathscr{D}_{\mathrm{rb}}$ is now a solution to the nonlinear functional integral equation given by

$$\begin{aligned} \mathscr{D}_{\mathrm{rb}}(t,v) &= \sum_{k\in\mathbb{Z}^d} \mathcal{E}(t,k)\,\frac{k\otimes k}{|k|^2}\,\Re e\int_0^\infty d\sigma\, e^{-\mathrm{i}(\omega(k)-k\cdot v)\sigma} A_{\mathrm{rb}}(\sigma,k,\mathscr{D}_{\mathrm{rb}}(\sigma,v)) \\ &= \sum_{k\in\mathbb{Z}^d} \mathcal{E}(t,k)\,\frac{k\otimes k}{|k|^2} R_{\mathrm{rb}}(\omega(k)-k\cdot v,k,\mathscr{D}_{\mathrm{rb}}). \end{aligned} \tag{144}$$

The resonance function R_{rb} can no more be seen as the Fourier transform (problem of convergence for negative time) of the time autocorrelation function A_{rb} but as the Laplace transform of it. The broadening of wave–particle resonance is produced by the term $k\otimes k : \mathscr{D}_{\mathrm{rb}}(\sigma,v)\sigma^3/3$, which can be seen an approximation of the ensemble-average mean square deviation from the mean of particle trajectories in the turbulent electric field (see Remark 14 for a rough but short explanation and Refs. 27 and 58 for a more rigorous and exhaustive one). Resonance broadening theory would certainly deserve a rigorous mathematical treatment, but it is out of the scope of this paper. This remark just aims at enlightening similarities and differences with the resonance broadening effect obtained in Corollary 2.

Remark 14. In this remark, we roughly explain where typical expression (143) comes from. Here, we give a short and simplified derivation of (143) just to give a flavor. A more involved and detailed derivation can be found in Refs. 27 and 58. In addition, without loss of generality, we consider the dimension $d = 1$ to simplify the calculation. In (141) or (144), the term $\exp(\mathrm{i}kv\sigma)$ can be seen as the result of approximating characteristic curves by the ballistic motion (like in the quasilinear theory), i.e., $X(\sigma;0,x,v) \simeq x + v\sigma$ and $V(\sigma;0,x,v) \simeq v$. Indeed, we observe that

$$\exp(\mathrm{i}kv\sigma) = \mathbb{E}[\exp(\mathrm{i}k\Delta X(\sigma))],$$

with $(\Delta X(\sigma),\Delta V(\sigma)) = (X(\sigma)-X(0),V(\sigma)-V(0)) = (v\sigma,v)$. Considering a higher approximation of the characteristic curves (in particular, an approximation that takes into account the diffusion of particles), we should add to the free-flow approximation a correction $(\delta X(\sigma),\delta V(\sigma))$ such that we have $X(\sigma;0,x,v) \simeq x + v\sigma + \delta X(\sigma)$ and $V(\sigma;0,x,v) = v + \delta V(\sigma)$. The autocorrelation function $\sigma \mapsto A_{\mathrm{rb}}(\sigma,k)$ is then defined by

$$A_{\rm rb}(\sigma,k) := \mathbb{E}[\exp(\mathrm{i}k\delta X(\sigma))] = \mathbb{E}\left[\exp\left(\mathrm{i}k\int_0^\sigma ds\,\delta V(s)\right)\right]. \tag{145}$$

We now suppose that $\delta V(\sigma)$ *is a Gaussian random function, with a probability distribution* $\mathbb{P}$ *given by*

$$\mathbb{P}(\delta V(\sigma)) = \frac{1}{\sqrt{2\pi\mathscr{D}(v)\sigma}}\exp\left(-\frac{(\delta V(\sigma))^2}{2\mathscr{D}(v)\sigma}\right),$$

where the standard deviation or the diffusion coefficient $\mathscr{D}(v)$ *is constant in time. Therefore, we have*

$$\mathbb{E}[\delta V(\sigma)] = 0, \quad \mathbb{E}[(\delta V(\sigma))^2] = \mathscr{D}(v)\sigma, \quad \text{and} \quad \mathbb{E}[\delta V(\sigma)\delta V(\sigma')] = \mathscr{D}(v)\min(\sigma,\sigma'), \tag{146}$$

which means that particle velocity follows a diffusion process characterized by the diffusion coefficient $\mathscr{D}$*. Since* $\delta V(\sigma)$ *is a Gaussian random function, using (146), we obtain from (145),*

$$\begin{aligned} A_{\rm rb}(\sigma,k) &= \mathbb{E}\left[\exp\left(\mathrm{i}k\int_0^\sigma \delta V(s)ds\right)\right] \\ &= \exp\left(-\frac{1}{2}k^2\int_0^\sigma ds\int_0^\sigma ds'\,\mathbb{E}[\delta V(s)\delta V(s')]\right) \\ &= \exp\left(-\frac{1}{2}k^2\int_0^\sigma ds\int_0^\sigma ds'\,\mathscr{D}(v)\min(s,s')\right) \\ &= \exp\left(-\frac{1}{6}k^2\mathscr{D}(v)\sigma^3\right), \end{aligned}$$

which gives, up to a mutiplicative constant in the exponential, the same result as (143).

2. Quasilinear theory: Infinite $\bar{\tau}$

Here, we show that we can retreive formally the diffusion matrix of the quasilinear theory[21,26,57] from the diffusion matrix (83) or (140) by taking the formal limit $\bar{\tau}\to+\infty$. For this, we choose an autocorrelation function $A_{\bar{\tau}}$ such that $A_{\bar{\tau}}\to 1$ a.e. as $\bar{\tau}\to+\infty$ [e.g., $A_{\bar{\tau}}(\sigma,k) = \mathbb{1}_{[-\bar{\tau},\bar{\tau}]}(\sigma)$]. Then, letting $\bar{\tau}$ go to infinity, we obtain in the sense of distributions

$$\lim_{\bar{\tau}\to+\infty}\int_0^{\bar{\tau}} d\sigma\, e^{-\mathrm{i}(\omega(k)-k\cdot v)\sigma}A_{\bar{\tau}}(\sigma,k) = \pi\delta(\omega(k)-k\cdot v)\ -\ \mathrm{i}\,\mathrm{p.v.}\left(\frac{1}{\omega(k)-k\cdot v}\right)$$

and

$$\lim_{\bar{\tau}\to+\infty} R_{\bar{\tau}}(\omega(k)-k\cdot v,k) = \pi\delta(\omega(k)-k\cdot v). \tag{147}$$

Therefore, the diffusion matrix (140) becomes the quasilinear diffusion matrix of plasma physics literature (see, e.g., Refs. 21 and 39),

$$\mathscr{D}_\infty(t,v) = \pi\sum_{k\in\mathbb{Z}^d}\mathcal{E}(t,k)\ \frac{k\otimes k}{|k|^2}\ \delta(\omega(k)-k\cdot v). \tag{148}$$

The diffusion matrix (148) seems not very regular, since it involves a sum of Dirac masses, namely, resonances $\delta(\omega(k)-k\cdot v)$. Then, well-posedness of the diffusion equation (84) with such a diffusion matrix remains an open issue at least for non-smooth distribution functions. We note that we can also recover the quasilinear diffusion matrix (148) from the diffusion matrix $\mathscr{D}_{\rm rb}$ of the resonance broadening theory (see Remark 13) by observing that $R_{\rm rb}(\omega(k)-k\cdot v,k,0) = \delta(\omega(k)-k\cdot v)$. From (143), this still corresponds to an infinite autocorrelation time $\bar{\tau}$. The limit $\bar{\tau}\to+\infty$ is a singular limit from different points of view:

1. When $\bar{\tau}\to\infty$, the autocorrelation matrix (138) is no more integrable with respect to correlation time σ but only locally integrable. This is the same for the autocorrelation matrix $\mathcal{R}_{\bar{\tau}}$ constructed in the Appendix. This loss of integrability entails a loss of regularity in the velocity variables for the diffusion matrix. This loss of regularity in velocity variables is even more striking when we observe the singular limit (147) for a smooth resonance function $R_{\bar{\tau}}$.
2. When $\bar{\tau}\to\infty$, hypothesis (H2) does not hold anymore. Indeed the stochastic electric field defined in Sec. IV A no longer satisfies a time decorrelation property since its decorrelation time tends to infinity. It is like falling back to the deterministic case.

When $\bar{\tau} \to \infty$, the autocorrelation time of particles tends to infinity and the time decorrelation of the electric field defined in Sec. IV A occurs at infinite time. This can be interpreted as follows: The electric field becomes deterministic and particles trajectories are almost straight lines. This seems consistent with the fact that the original derivation of the QL theory performed by physicists[26,57] is deterministic. Indeed, this deterministic derivation is based on two main ingredients. First, the wave–particle interaction is assumed perturbative, and the perturbed dynamics of particles is approximated by the free flow or the ballistic motion. Second, all nonlinear wave–wave interactions, except for their effect on the space-averaged distribution function, are neglected. After the original 1962 derivation, which is deterministic, other derivations of the QL theory (see, e.g., Refs. 3, 5, and 21) appeal to some statistical arguments and decorrelation hypotheses to establish the QL diffusion. Therefore, considering quasilinear theory as a probabilistic or deterministic theory remains an open question.

Finally, we note that the resonance broadening theory[27,58] is actually a statistical (probabilistic) theory of the Vlasov equation and does not have a deterministic counterpart in the plasma physics literature. Nevertheless, in Sec. III B 2 for the deterministic case, we have been able to introduce a finite autocorrelation time of particles $\bar{\tau}$ and to derive formally a diffusion matrix that is consistent with the quasilinear one in the limit $\bar{\tau} \to \infty$.

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APPENDIX: AN EXAMPLE OF A RANDOM FIELD *E*

In this appendix, we construct an example of random electric field $E = E(t,\tau,x)$ that satisfies hypotheses (H1)–(H4) of Sec. IV A. For this, we are inspired by the Example 2 in Ref. 50. Let r and ϱ be positive real numbers such that $\varrho \geq 1/r$ and $r \geq 1$. We set $\bar{\tau} = 1/r + \varrho$. With this decomposition of the autocorrelation time $\bar{\tau}$, we can choose $\bar{\tau}$ as small as we want by taking finite but large r and choose also $\bar{\tau}$ as large as we want by taking any fixed r and ϱ large enough. Let $(T_k^n, X_k^n) \in \mathbb{R}^{1+d}$ with $(n,k) \in \mathbb{Z} \times \mathbb{Z}^d$ be independent random variables equidistributed in

$$\left\{\frac{n}{r} + \left[-\frac{1}{2r}, \frac{1}{2r}\right]\right\} \times \left\{k + [-1/2, 1/2]^d\right\}.$$

We consider also other independent random variables α_k^n, with $(n,k) \in \mathbb{Z} \times \mathbb{Z}^d$, such that $\mathbb{E}[\alpha_k^n] = 0$ and $\mathbb{E}[(\alpha_k^n)^2] = 1$. Let $\eta \in \mathscr{C}_c^\infty(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d; \mathbb{R})$ be a real scalar function that is compactly supported in $\mathbb{R}^+ \times [-\varrho/2, \varrho/2] \times \mathbb{R}^d$. We define the random function $\mathcal{E} = \mathcal{E}(t,\tau,x)$ by

$$\mathcal{E}(t,\tau,x) = \sum_{(n,k)\in\mathbb{Z}^{1+d}} \alpha_k^n \eta(t, \tau - T_k^n, x - X_k^n).$$

Since η has a compact support, the sum defining $\mathcal{E}$ in this equation is finite. Obviously, $\mathcal{E} \in \mathscr{C}_c^\infty(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d; \mathbb{R})$ and $\mathbb{E}[\mathcal{E}] = 0$, which means that hypotheses (H1) and (H4) are satisfied. For a fixed $\tau \in \mathbb{R}$, the function $\mathcal{E}(t,\tau,x)$ depends only on $(T_l^n, X_l^n, \alpha_l^n)$ with

$$n \in \left[\tau - \frac{1}{2r} - \frac{\varrho}{2}, \tau + \frac{1}{2r} + \frac{\varrho}{2}\right].$$

Similarly, we define

$$\mathcal{E}'(s,\sigma,y) = \sum_{(n,k)\in\mathbb{Z}^{1+d}} \alpha_k^n \eta'(s, \sigma - T_k^n, y - X_k^n),$$

with $\eta' \in \mathscr{C}_c^\infty(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d; \mathbb{R})$ being a real scalar function, which is also compactly supported in $\mathbb{R}^+ \times [-\varrho/2, \varrho/2] \times \mathbb{R}^d$. For a fixed $\sigma \in \mathbb{R}$, the function $\mathcal{E}'(s,\sigma,x)$ depends only on $(T_k^m, X_k^m, \alpha_k^m)$ with

$$m \in \left[\sigma - \frac{1}{2r} - \frac{\varrho}{2}, \sigma + \frac{1}{2r} + \frac{\varrho}{2}\right].$$

Then, as soon as $|\tau - \sigma| \geq 1/r + \varrho =: \bar{\tau}$, the random functions $\mathcal{E}(t,\tau,x)$ and $\mathcal{E}'(s,\sigma,y)$ are independent, which means that hypothesis (H2) is satisfied. It remains to show hypothesis (H3). For this, we estimate $\mathbb{E}[\mathcal{E}(t,\tau,x)\mathcal{E}'(s,\sigma,y)]$ as follows: Using the independence of random variables (T_k^n, X_k^n) and α_k^n and using $\mathbb{E}[\alpha_k^n \alpha_l^m] = \delta_{mn}\delta_{kl}$, we obtain

$$\begin{aligned}
\mathbb{E}[\mathcal{E}(t,\tau,x)\mathcal{E}'(s,\sigma,y)] &= \sum_{(n,k)\in\mathbb{Z}^{1+d}}\sum_{(m,l)\in\mathbb{Z}^{1+d}}\mathbb{E}[\alpha_k^n\alpha_l^m] \\
&\quad \mathbb{E}[\eta(t,\tau-T_k^n,x-X_k^n)\eta'(s,\sigma-T_l^m,y-X_l^m)] \\
&= \sum_{(n,k)\in\mathbb{Z}^{1+d}}\mathbb{E}[(\alpha_k^n)^2]\mathbb{E}[\eta(t,\tau-T_k^n,x-X_k^n)\eta'(s,\sigma-T_k^n,y-X_k^n)] \\
&= \sum_{(n,k)\in\mathbb{Z}^{1+d}}\int_{[-1/2,1/2]^d}dz\int_{-\frac{1}{2r}}^{\frac{1}{2r}}d\theta \\
&\quad \eta(t,\tau-n/r-\theta,x-k-z)\eta'(s,\sigma-n/r-\theta,y-k-z) \\
&= \sum_{(n,k)\in\mathbb{Z}^{1+d}}\int_{k+[-1/2,1/2]^d}dz\int_{(n-1/2)/r}^{(n+1/2)/r}d\theta \\
&\quad \eta(t,\tau-\theta,x-z)\eta'(s,\sigma-\theta,y-z) \\
&= \int_{\mathbb{R}^d}dz\int_{\mathbb{R}}d\theta\,\eta(t,\tau-\theta,x-z)\eta'(s,\sigma-\theta,y-z) \\
&= \int_{\mathbb{R}^d}dz\int_{\mathbb{R}}d\theta\,\eta(t,\tau-\sigma+\theta,x-y+z)\eta'(s,\theta,z) \\
&=: \widetilde{\mathcal{R}}_{\eta,\eta'}(t,s,\tau-\sigma,x-y).
\end{aligned}$$

We then obtain $\widetilde{\mathcal{R}}_{\eta,\eta'}\in\mathscr{C}_c^\infty(\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^d;\mathbb{R})$. To construct the autocorrelation matrix $(\mathcal{R}_{\bar{\tau}\,ij})_{(i,j)}$, we first choose real scalar functions $\eta_i\in\mathscr{C}_c^\infty(\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^d;\mathbb{R})$ for $i\in\{1,\ldots,d\}$, which are compactly supported in $\mathbb{R}^+\times[-\varrho/2,\varrho/2]\times\mathbb{R}^d$. Second, we define the i-th component of the random electric field E by

$$E_i(t,\tau,x) = \sum_{(n,k)\in\mathbb{Z}^{1+d}}\alpha_k^n\eta_i(t,\tau-T_k^n,x-X_k^n).$$

Finally, the autocorrelation matrix $\mathcal{R}_{\bar{\tau}}$ is defined by

$$\mathcal{R}_{\bar{\tau}\,ij}(t,s,\tau-\sigma,x-y) := \widetilde{\mathcal{R}}_{\eta_i,\eta_j}(t,s,\tau-\sigma,x-y).$$

Another way to obtain the autocorrelation matrix $\mathcal{R}_{\bar{\tau}}$ is to take the gradient of $\widetilde{\mathcal{R}}_{\eta,\eta'}(t,s,\tau-\sigma,x-y)$ with respect to the variables x and y, i.e.,

$$\begin{aligned}
\mathcal{R}_{\bar{\tau}\,ij}(t,s,\tau-\sigma,x-y) &:= \partial_{x_i}\partial_{y_j}\widetilde{\mathcal{R}}_{\eta,\eta'}(t,s,\tau-\sigma,x-y) = \widetilde{\mathcal{R}}_{\partial_{x_i}\eta,\partial_{y_j}\eta'}(t,s,\tau-\sigma,x-y) \\
&= \mathbb{E}[\partial_{x_i}\mathcal{E}(t,\tau,x)\partial_{y_j}\mathcal{E}'(s,\sigma,y)].
\end{aligned}$$

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

REFERENCES

[1] R. E. Aamodt, "Test waves in weakly turbulent plasmas," Phys. Fluids **10**, 1245–1250 (1967).
[2] J. C. Adam, G. Laval, and D. Pesme, "Reconsideration of quasilinear theory," Phys. Rev. Lett. **43**, 1671–1675 (1979).
[3] A. Akhiezer, I. Akhiezer, R. Polovin, A. Sitenko, and K. Stepanov, *Plasma Electrodynamics*, Linear Theory Vol. I and Non-Linear Theory and Fluctuations Vol. II (Pergamon Press, 1975).
[4] A. Arsenev, "Global existence of a weak solution of Vlasov's system of equations," U.S.S.R. Comput. Math. Math. Phys. **15**, 131–141 (1975).
[5] R. Balescu, *Aspects of Anomalous Transport in Plasmas* (Institut of Physics Publishing, 2005).
[6] C. Bardos, F. Golse, and J.-F. Colonna, "Diffusion approximation and hyperbolic automorphisms of the torus," Physica D **104**, 32–60 (1997).
[7] P. Bechouche, F. Poupaud, and J. Soler, "Quantum transport and Boltzmann operators," J. Stat. Phys. **122**, 417–436 (2006).
[8] D. Bénisti and D. F. Escande, "Origin of diffusion in Hamiltonian dynamics," Phys. Plasmas **4**, 1576–1581 (1997).
[9] D. Bénisti and D. F. Escande, "Finite range of large perturbations in Hamiltonian dynamics," J. Stat. Phys. **92**, 909–972 (1998).

[10] N. Besse, Y. Elskens, D. F. Escande, and P. Bertrand, "Validity of quasilinear theory: Refutations and new numerical confirmation," Plasma Phys. Controlled Fusion **53**, 025012 (2011).
[11] N. Besse, Y. Elskens, D. F. Escande, and P. Bertrand, "On the validity of quasilinear theory," in *Proceeding 38th EPS Conference on Controlled Fusion and Plasma Physics* (European Physical Society, Strasbourg, 2011), p. P2.009, http://ocs.ciemat.es/EPS2011PAP/pdf/P2.009.pdf; accessed 3 April 2018.
[12] F. Bouchut, F. Golse, and M. Pulvirenti, *Kinetic Equations and Asymptotic Theory*, Series on Applied Mathematics (Gauthiers-Villars, 2000).
[13] N. Bourbaki, *Elements of Mathematics: General Topology* (Springer, 1989).
[14] T. Boutros-Ghali and T. H. Dupree, "Theory of two-point correlation function in Vlasov plasma," Phys. Fluids **24**, 1839–1858 (1981).
[15] M. Brassart, *Limite Semi-Classique de Transformées de Wigner Dans des Milieux Périodiques ou Aléatoires* (Thèse de l'université de Nice Sophia-Antipolis, 2002).
[16] O. Buneman, "Dissipation of currents in ionized media," Phys. Rev. **115**, 503–517 (1959).
[17] E. Caglioti and C. Maffei, "Time asymptotics for solutions of Vlasov–Poisson equation in a circle," J. Stat. Phys. **92**, 301–323 (1998).
[18] J. R. Cary, D. F. Escande, and A. D. Verga, "Non quasilinear diffusion far from chaotic threshold," Phys. Rev. Lett. **65**, 3132–3135 (1990).
[19] J. R. Cary, I. Doxas, D. F. Escande, and A. D. Verga, "Enhancement of the velocity diffusion in longitudinal plasma turbulence," Phys. Fluids B **4**, 2062–2069 (1992).
[20] F. Castella, P. Degond, and T. Goudon, "Asymptotic problems for wave–particle interactions: Quantum and classical models," Nonlinearity **20**, 1677–1720 (2007).
[21] R. C. Davidson, *Methods in Nonlinear Plasma Theory* (Academic Press, 1972).
[22] R. J. DiPerna and P.-L. Lions, "Solutions globales d'équations du type Vlasov–Poisson," C. R. Acad. Sci. Paris, Sér. I **307**, 655–658 (1988).
[23] R. J. DiPerna and P.-L. Lions, "Global weak solutions of kinetic equations," Rend. Semin. Mat. Univ. Politecn. Torino **46**, 259–288 (1988).
[24] R. J. DiPerna and P. L. Lions, "Global weak solutions of Vlasov–Maxwell systems," Commun. Pure Appl. Math. **42**, 729–757 (1989).
[25] I. Doxas and J. R. Cary, "Numerical observation of turbulence enhanced growth rates," Phys. Plasmas **4**, 2509–2518 (1997).
[26] W. E. Drummond and D. Pines, "Non-linear stability of plasma oscillations," Nucl. Fusion Suppl. **3**, 1049–1057 (1962).
[27] T. H. Dupree, "A perturbation theory for strong plasma oscillations," Phys. Fluids **9**, 1773–1782 (1966).
[28] T. H. Dupree, "Theory of phase space density granulation in plasma," Phys. Fluids **15**, 334–344 (1972).
[29] Y. Elskens and E. Pardoux, "Diffusion limit for many particles in a periodic stochastic acceleration field," Ann. Appl. Probab. **20**, 2022–2039 (2010).
[30] Y. Elskens, "Gaussian convergence for stochastic acceleration of *N* particles in the dense spectrum limit," J. Stat. Phys. **148**, 591–605 (2012).
[31] Y. Elskens and D. Escande, *Microscopic Dynamics of Plasmas and Chaos* (Institut of Physics, 2003).
[32] D. F. Escande and Y. Elskens, "Proof of quasilinear equations in the chaotic regime of the weak warm beam instability," Phys. Lett. A **302**, 110–119 (2002).
[33] D. F. Escande and Y. Elskens, "Proof of quasilinear equations in the strongly nonlinear regime of the weak warm beam instability," Phys. Plasmas **10**, 1588–1594 (2003).
[34] D. F. Escande, "Wave–particle interaction in plasmas: A qualitative approach," in *Long-Range Interacting Systems*, edited by T. Dauxois, S. Ruffo, and L. F. Cugliandolo (Oxford University Press, 2010), pp. 817–836.
[35] D. F. Escande, "From thermonuclear fusion to Hamiltonian chaos," Eur. Phys. J. H **43**, 397–420 (2018).
[36] A. A. Galeev, R. Z. Sagdeev, V. D. Shapiro, and V. I. Shevchenko, "Is renormalization necessary in the quasi-linear theory of Langmuir oscillations?," Sov. Phys. JETP **52**, 1095–1099 (1980).
[37] T. Goudon and F. Poupaud, "Homogenization of transport equations: A simple PDE approach to the Kubo formula," Bull. Sci. Math. **131**, 72–88 (2007).
[38] D. A. Hartmann, C. F. Driscoll, T. M. O'Neil, and V. D. Shapiro, "Measurements of the warm beam instability," Phys. Plasmas **2**, 654–677 (1995).
[39] N. A. Krall and A. W. Trivelpiece, *Principles of Plasma Physics* (McGraw-Hill, 1973).
[40] G. Laval and D. Pesme D, "Breakdown of quasilinear theory for incoherent 1-D Langmuir waves," Phys. Fluids **26**, 52–65 (1983).
[41] G. Laval and D. Pesme, "Inconsistency of quasilinear theory," Phys. Fluids **26**, 66–68 (1983).
[42] G. Laval and D. Pesme, "Self-consistency effects in quasilinear theory: A model for turbulent trapping," Phys. Rev. Lett. **53**, 270–273 (1984).
[43] G. Laval and D. Pesme, "Controversies about quasi-linear theory," Plasma Phys. Controlled Fusion **41**, A239–A246 (1999).
[44] G. Laval, D. Pesme, and J.-C. Adam, "Wave–particle and wave–wave interactions in hot plasmas: A French historical point of view," Eur. Phys. J. H **43**, 421–458 (2018).
[45] Y.-M. Liang and P. H. Diamond, "Weak turbulence theory of Langmuir waves: A reconsideration of the validity of quasilinear theory," Comments Plasma Phys. Controlled Fusion **15**, 139–149 (1993).
[46] Y. M. Liang and P. H. Diamond, "Revisiting the validity of quasilinear theory," Phys. Fluids B **5**, 4333–4340 (1993).
[47] G. Loeper and A. Vasseur, "Electric turbulence in a plasma subject to a strong magnetic field," Asymptotic Anal. **40**, 51–65 (2004).
[48] C. Mouhot and C. Villani, "On Landau damping," Acta Math. **207**, 29–201 (2010).
[49] O. Penrose, "Electrostatic instabilities of a uniform non-Maxwellian plasma," Phys. Fluids **3**, 258–265 (1960).
[50] F. Poupaud and A. Vasseur, "Classical and quantum transport in random media," J. Math. Pures Appl. **82**, 711–748 (2003).
[51] J. C. Price, "Kinetic theory of inhomogeneous systems," Phys. Fluids **12**, 593–606 (1969).
[52] C. Roberson and K. W. Gentle, "Experimental test of the quasilinear theory of the gentle bump instability," Phys. Fluids **14**, 2462–2469 (1971).
[53] L. Saint-Raymond, *Hydrodynamic Limits of the Boltzmann Equation*, Lecture Notes in Mathematics Vol. 1971 (Springer, 2009).
[54] V. D. Shapiro and R. Z. Sagdeev, "Nonlinear wave–particle interaction and conditions for applicability of quasilinear theory," Phys. Rep. **283**, 49–71 (1997).
[55] J. Simon, "Compact sets in the space $L^p(0, T; B)$," Ann. Math. Pura. Appl. **146**, 65–96 (1987).
[56] S. I. Tsunoda, F. Doveil, and J. H. Malmberg, "Experimental test of quasilinear theory," Phys. Fluids B **3**, 2747–2757 (1991).
[57] A. A. Vedenov, E. P. Velikhov, and R. Z. Sagdeev, "Quasilinear theory of plasma oscillations," Nucl. Fusion Suppl. **2**, 465–475 (1962).
[58] J. Weinstock, "Formulation of a statistical theory of strong plasma turbulence," Phys. Fluids **12**, 1045–1058 (1969).
[59] S. Willard, *General Topology* (Addison Wesley Publishing Company, 1970).
[60] K. Yosida, *Functional Analysis* (Springer, 1980).

[61] In fact, this diffusion coefficient is a formal approximation for small γ of the quasilinear diffusion coefficient $D_{QL}(t,v) = \sum_{k\in\mathbb{Z}}|E(t,k)|^2 \frac{\gamma(t,k)}{(\omega(t,k)-kv)^2+\gamma^2(t,k)}$ in the resonant region of velocity space that corresponds to particle velocities satisfying $\omega(t,k) - kv = 0$.